Mathematics: Modeling our World
(ARISE)

Level: 9-12

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Publisher: South-Western Educational Publishing

Review Materials: Course 1, 2, and 3 are published. Course 4 is in preparation. Annotated Teacher=s Editions and Teacher Resource packages are available for each course. A motivational video accompanied by a brief written guide is part of the instructional materials for each year. Also available is a CD-ROM with software written specifically for the curriculum as well as data sets, satellite images and spreadsheet templates. In addition, other software may be downloaded free of charge from the Internet. A complete Solutions Manual is available for each year.

Format/description: This is a complete four-year secondary school mathematics curriculum for all students. Mathematical modeling, including the modeling process itself, is a central focus throughout the curriculum. In addition, a major goal of the curriculum is the development of higher order thinking skills rather than just Agetting answers.@ Such skills include the ability to transfer ideas from one context to another and make connections between mathematical material as well as material in disciplines outside mathematics. This is an integrated curriculum in that traditional disciplines of mathematics are tightly woven together; the boundaries between them are blurred, and there is a successful attempt to unify the various content areas of mathematics. Each grade level contains seven or eight *units*. Each unit develops mathematical content material by focusing on a central unit context, such as animation or secret codes. Additional applications of the mathematical content are often included within the unit. Each unit begins with students viewing a video segment (provided) designed to be a motivator and for setting the context for the unit. Also, the student materials begin each unit with a *unit opener* which is intended to pique student interest and set the tone for the unit. Each unit is divided into four to seven *lessons*. Individual lessons typically take several days to complete. Each lesson begins with a *lesson opener* which lists key concepts presented in the unit and preparation reading which focuses the context for the lesson. Moreover, this section of the lesson raises issues and encourages students to ask questions concerning the unit context. Following the unit opener, each lesson is usually divided into several *activities*. Each activity includes work to be done by a pair or group of students as well as work designated as *individual work*. Student materials include *unit summaries* which contain work intended for reviewing each unit of material and Amathematical summaries@ which contain overviews of the mathematical content of the unit as well as a glossary of mathematical terms contained in each unit.

The annotated Teacher Edition for each Course contains a unit overview listing the major mathematical themes, where they appear in the unit, a verbal overview of the unit context, relations of the unit to disciplines outside mathematics, list of materials needed, and teaching suggestions for presenting the context as well as additional background information. For each lesson, it contains suggested pacing, a description of the lesson purpose, list of materials needed, and additional suggestions for approaches to the lesson. For individual activities, it provides notes to help guide students through the materials, suggestions for where to include additional material and reduced transparency masters as well as suggestions on where to use them. For the unit summaries, it contains suggestions on how to review skills and concepts. The Teacher Resource package includes suggestions on how to use the video segments and lesson by lesson supplemental information and suggestions. It also provides additional background reading and software information. It includes handouts, supplemental activities, assessment problems, and transparencies with suggestions for their use. In addition, it contains Unit Projects which typically draw on all concepts in the unit. A collection of assessment problems is also included.

Pedagogy: The curriculum delivery focus is on student learning. All materials are written in a way that stresses active student participation. Students spend their class time working in pairs or groups and individually, as well as discussing material as a whole class. Students are *consistently* asked to explain their reasoning and support their conclusions, as well as focus on the process of mathematical modeling. Differences in interest and performance are accommodated through the individual work that varies in difficulty and through the use of the supplementary activities and projects. See the various teacher materials that accompany the curriculum for more detail.

Technology: The use of technology is a fundamental component of the curriculum. That is, graphing calculators, spreadsheets, geometric drawing utilities and software written specifically for use with this curriculum are used frequently. Not all technology is necessary for every unit. Moreover, it may be possible to modify some units when technology is unavailable. However, graphing calculators are essential. Technology is used for purposes of tedious calculation, for simulation, and for stressing relationships between various representations of concepts. It does not replace understanding with the senseless execution of a sequence of buttons.

Assessment: A wide variety of assessment problems including many which are open-ended are provided in the Teacher Resource package. Suggestions for their placement are contained in the Teacher's Edition of the student materials so that assessment may be embedded.

Content Overview: The first four NCTM Standards: Mathematics as Problem Solving, as Communication, as Reasoning, and Mathematical Connections are addressed heavily in every unit. Students are consistently asked thought-provoking questions and required to explain their reasoning, especially in the context of the process of mathematical modeling. Many questions are open-ended and/or require creative solutions. Students develop the ability to make conjectures, design criteria, and establish assumptions about a model. Referring to previous contexts often makes connections between the units. In doing so, students can see the value of mathematical abstraction in the sense that mathematical ideas can be useful in many contexts. In addition, the varying contexts serve to illuminate advantages of *various* attributes of a mathematical concept. For example, students are continually asked to describe the advantages and disadvantages of different representations of a concept in a context. The contexts have been carefully chosen to appeal to the students without diminishing the importance and usefulness of the mathematics; indeed, the contexts heighten the usefulness of the concepts in our present era. Students become acquainted with mathematical definitions and terminology in the course of solving problems.

The curriculum may be described using five strands: **algebra/number/function, geometry/trigonometry, statistics/probability, discrete mathematics, and logic/reasoning.** We reiterate that this is an integrated curriculum and one of the advantages of such an approach is to develop mathematics as a unified body of knowledge. These strands are used for descriptive purposes only. Indeed, some content may seem to be misplaced in the description below. However, when concepts are developed together, it is sometimes very inappropriate and defeating to sever their linkages. Furthermore, some topics could be presented under more than one heading. However, for the most part, the decision was made to place them under only one heading. For ease of reading, the description below does not necessarily present the topics in the order in which they appear within each grade. Finally, some review and additional practice occurs often in the material. For example, solving quadratic equations is first encountered in Year 1. However, it is also practiced in the last unit of Year 2. Such reviews are not mentioned in the following description.

Course 1

Algebra/number/function: Students will understand the concept of a function as a way of Atransforming@ objects. They will also understand the concept of function as a set of ordered pairs and as a method used to create a single output from a given input. They will be able to represent functions using words, tables, graphs, symbolic expressions, including the use of the notation $f(x)$, and arrow diagrams. They will know some advantages and disadvantages of each representation. They will be able to use matrix addition as a way of transforming several inputs at once. They will understand the use of the function concept to express a pattern. They will be able to determine the domain and range of a function in various contexts. They will understand the meaning of and be able to create Aequivalent expressions@ as illustrated, for example, by the distributive law. They will understand the value of equivalent expressions in showing, for example, that multi-step descriptions may be equivalent to less complicated or shorter processes. They will understand and be able to use the concept of an inverse of a function and the role the order of operations plays in describing the inverse of certain functions. They will understand the role of the concept of function in terms of to view coding information. In fact, they will understand and be able to model and represent, in many ways, several types of coding including the Ashift cipher@ (translation) the Astretch@ cipher (dilation), a combination of the latter two (forming a linear function), and a Akeyword@ cipher using matrices. They will also model their own codes to satisfy certain criteria. They will be able to use their knowledge of these codes, frequency distributions of letters, and trial and error to Abreak@ some unknown codes. Students will understand and be able to use two forms of equations of lines: the *slope-intercept* form: $y=mx+b$; and the *point-slope* form: $y=m(x-h)+k$ (or, equivalently, $y-k=m(x-h)$). They will be able to interpret the concept of slope geometrically (as the Atilt@ of the line), proportionally as $\Delta y/\Delta x$, and contextually (in words). They will understand the relationship between the sign of the slope and the tilt of the line and they will know that horizontal lines have zero slope. They will be able to interpret the concept of y-intercept geometrically and contextually. They will know how changing the values of *m* and *b* in the slope-intercept form of the equation of a line affect the graph of the line. They will be able to move freely from linear graph to linear equation and from linear equation to linear graph. They will understand that *equivalent equations* refer to equations that have the same solutions. They will have algebraic and geometric strategies for determining whether or not two linear equations are equivalent. Students will model motion, particularly linear motion, in one and two dimensions. They will understand and be able to use the concept of (constant) *rate of change* in several contexts, including velocity. They will be able to associate the terms rate of change with the slope of a line describing *displacement*. They will understand that modeling motion requires a Areference system,@ such as a coordinate plane. They will be able to model motion of a point in the plane using a discrete model with *recurrence relations* (current location = previous location + displacement). They will also be able to model motion using continuous models involving parametric equations. They will be able to compare discrete and continuous models. They will be able to describe the path of an object, which is determined by linear parametric equations, by using at least two strategies to eliminate the parameter and obtain a linear function of the form y = mx +b. They will be able to create Atime series@ graphs for the motion of a point described by parametric equations. They will be able to explain the role of the Acontrol numbers@ *a, b, c,* and *d*, in the parametric equations $x = ax + b$ and $y = cx + d$. They will be able to move from Aadditive@ recurrence relations (such as $p_{current} = p_{prior} + 28$) to *closed-form* linear functions and vice versa. They will have at least two strategies for solving a system of two linear equations. They will be able to use matrices, matrix addition and scalar multiplication to model the movement of several points at once. At least one of these models will combine matrices and parametric equations. In addition, they will be able to represent growth and decay in time series graphs using models involving additive recurrence relations and models involving Amultiplicative@ recurrence relations (such as $p_{current} = (1+r)p_{prior}$). They will be able to move from multiplicative recurrence relations to closed-form exponential functions. They will be able to explain the role of the control numbers *a* and *b* in equations of the form $y = ab^x$ and $y=a(1+b)^x$ with regard to the shape of the graph of this equation and in contextual situations. They will know, be able to use, and be able to explain the difference between the terms *relative rate of growth* and *growth factor*. They will recognize that exponential functions are usually valuable in modeling situations that contain an assumption that the amount of change in a quantity is proportional to the amount present. They also will have some strategies for spotting exponential relationships in data. Moreover, they will have algebraic strategies for fitting exponential functions to two data points in the plane. They will have developed contextual (combinatorial) explanations and interpretations for the laws of exponents. Students will have conducted *sensitivity analysis* in both models involving linear functions and models involving exponential functions. They also will have explored how scale changes and translation shifts affect the functions both algebraically and graphically in models involving linear or exponential growth. In general, students will be able to discuss advantages and disadvantages of closed-form and recurrence relations. In addition to analyzing situations using closed-form and recurrence relations, students will have been introduced to the additional tool of *simulation* and used it in at least one scenario involving a probabilistic model which goes beyond deterministic algebraic analysis. Students will understand usual functional notation, such as p(1988) and conjoined inequalities, such as 15#p(1988)#20. Students will be able to compare two quadratic expressions using several specific test values, graphs or tables, and symbolic manipulation to determine whether or not they are equivalent. They will be able to convert the general form of a quadratic function ($y=ax^2+bx+c$) to vertex form ($y=a(x+h)^2+k$), and vice versa. They will be able to explain the role of the control numbers *a*, *b*, and *c* in the general form of a quadratic function by examining transformations (vertical and horizontal translations and Aflattening/elongating@). Students will explore strategies for solving a quadratic equation. In particular, they will utilize a Azoom-and-trace@ approach with a calculator on the graph of the quadratic function. They will also consider zooming in on a spreadsheet table to look at a more confined range of values. In addition, they will solve quadratic equations in vertex form using arrow diagrams. They will also solve quadratic equations symbolically by completing the square and using the quadratic formula.

Geometry/trigonometry: Students will understand and be able to use the concepts of *scale* and *conversion* of units. They will also understand the term *scale factor*. They will be able to express a conversion symbolically as $y = cx$. They will understand *similarity* between two one dimensional or two two-dimensional figures in terms of a comparison of the distance between any pair of points in one figure and the distance between the corresponding points in the second figure. That is, if two figures are similar, there is a constant s such that given any pair of points in the first figure, $\Delta R = s\Delta M$, where ΔM is the distance between the two points in the first figure and ΔR is the distance between the corresponding points. They will know that the latter constant s is a scale factor. They will also understand that figures are similar if the relative size between objects in one figure is the same as the relative size between the corresponding objects in the second figure. In particular, in similar figures, ratios of lengths of objects are preserved. They will be able to use a (non-vertical) straight line in the plane to transform a straight line segment on the x-axis into a similar segment on the y-axis by ›reflecting the line segment through the line.@ They will understand that the scale factor for this transformation is determined by the tilt of the line (and vice versa). They will understand what *dilations* and *translations* are geometrically and they will have developed experimental evidence of the fact that dilations of figures produce similar figures (but not all similar figures Acome from@ the process of dilation). They will understand and explain why, for example, Azooming in@ on a computer screen or a graphing calculator screen is a dilation. They will be able to represent dilations and translations geometrically, using words, using tables, using arrow diagrams, and using symbolic expressions for the transformation of coordinates when the dilation of a two dimensional object occurs in a coordinate plane. They will know that a dilation is characterized by a fixed center and a Amagnitude@ (scale factor). Given two-dimensional figures which are related by a dilation, students will be able to find the dilation or given a fixed center, they will be able to construct the composition (the word is not used, yet) of a dilation followed by a translation which relate the two figures. They will understand the importance of precision (and of not rounding intermediate calculations) in some applications of Ascaling.@ Students will be able to approximate areas in irregularly shaped figures using at least five strategies: circumscribed/inscribed rectangles, counting squares on a grid, the Adot@ method (modified square counting), a Monte Carlo method of square counting, and, for digitized images, pixel counting. They will be able to explain advantages and disadvantages of each. They will understand the limitations of Adigitizing@ real images (pixels represent infinitely many points, for example).

Probability/statistics: Students will explore the use of the *mean* as a predictor in one-dimensional data sets. Students will have some strategies for looking for associations between two variables. Such strategies will include the use of dot plots (line plots), scatterplots, and bar graphs. They will be able to describe some associations between two variables in terms of form (such as linear), direction (such as positive or negative slope), and strength. Students will have developed strategies with regard to when to suggest linear models and how to test such linear models between two variables based on data collected for the two variables. They will be able to use such models to make predictions and use mathematical concepts to explain whether or not they feel such predictions are reasonable. In particular, they will investigate criteria for determining a line that Abest fits@ the data on a scatterplot. Several candidates for such lines will be investigated. These candidates include but are not limited to the regression line (determined by least squares analysis). Students may suggest their own procedure for selecting linear candidates. They will investigate criteria that support a strong linear relationship between two variables. Such criteria will include, but is not limited to, looking for certain random patterns exhibited by *prediction errors* in a residual plot together with the sum of these errors. Standard (Pearson=s) correlation coefficients may or may not be studied. But, students will investigate how variability (spread) in the data affects precision in conclusions and understand that precision is also dependent on the quality of the data. In addition, students may develop their own criteria to be used to support a linear relationship. They will also investigate examples concerning when *not* to use scatterplots to help determine relationships between variables. They will look at relationships between two variables in terms of an *explanatory variable* and a *response variable*. In addition, students will be able to investigate and explain effects of outliers on the regression line. Students will investigate sampling to determine a population trait. They will see that variability in samples is linked to sample size and that the larger the representative sample size, the less the (relative) variability. They will use several methods to collect data, including simulation (with manipulatives and technology). They will explore several issues of imperfect testing including sensitive questions (possibly resulting in dishonest responses) in subjective surveys and false positives (and negatives) in objective tests. They will know, be able to use, and have developed algebraic relationships and strategies to analyze information collected using Warner=s randomized-response technique to mask responses in sensitive question surveys. They will be able to use tables, graphs, and glyph diagrams to represent data. They will be able to use Atwo-way@ tables and percentages to examine relationships between two characteristics in a population. Students will be able to evaluate and compare linear, exponential, and quadratic models of data by examining residuals. They will be able to perform linear, exponential, and quadratic regression using the calculator. They will be able to distinguish any differences between the domain of a linear, exponential or quadratic model and the domain of the situation it represents.

Students will interpret probability as the long-term frequency of the occurrence of an event. They will know and be able to use the terms *mutually exclusive* events. The will understand two events as *complementary* if, under given circumstances, at least one of the two events must happen, but both cannot happen at the same time. They will understand the probabilistic relationship between complementary events. They will understand conditional probability in terms of a restricted sample space (term is not used). They will understand that two events are *independent* if the unconditioned probability of one event is the same as the probability of the same event conditioned by the other event. They will be able to use tables, area models, and tree diagrams to represent probabilities. (Most tree diagrams are binary trees. However, some situations involve more splitting.) Students will see that relationships between events often translate into computational tools. They will have developed a linear model relating the fraction of the population with a trait and the fraction of positive test results obtained from an objective test. They will be able to use this model to estimate the percentage of a population with a trait given the percentage of positive test results (and vice versa). They will be able to explain the meaning of the slope and the y-intercept in the context for this model. They will have developed knowledge that this model depends on the probability that the test correctly detects the trait (p) and the probability correctly reports the absence of the trait (r). Furthermore, they will have developed a model involving a rational function (in this case, the quotient of two linear functions) relating the effectiveness of the objective test and the actual percentage of the population that has the trait. They will have realized that the effectiveness of the test increases as the percentage of the population with the trait increases. Students will know and be able to use a strategy for determining the likelihood that an individual has a trait if an objective test for that individual is positive. The strategy is Bayesian analysis (that term is not used and the strategy is not formally stated). Students will be able to determine the expected value for a given situation and interpret its significance.

Discrete mathematics: Students will have explored and understand several voting methods and their flaws. In particular they will have analyzed plurality voting, the runoff method, the (Borda) point system, the sequential runoff method, the Condorcet method, and approval voting. They may have proposed additional alternatives. They will also have examined *insincere voting* in conjunction with some of the aforementioned voting methods. They will be able to examine such voting methods using *preference diagrams* and *runoff diagrams* (using digraphs). They will also be able to use matrices for organizing data and bookkeeping with regard to voting scenarios. They will be able to examine voting situations using actual numbers and percentages and be able to compare these approaches. Students will be able to model motion by implementing a discrete model with a computer program (at, least using the resident language of a graphing calculator). They will understand and be able to use some of the basics of programming such as loops, iteration, counters, and accumulators.

Mathematical reasoning/modeling/logic: Students will develop a deep sense of the mathematical modeling process throughout the curriculum. They will know and be able to use a modeling process cyclically built on four steps: **clearly identify the situation**: pose a well-formed question; **simplify the situation**: list the key features you wish to consider and assumptions on which the model will rest using as simple a model as possible and decide on what to measure for evidence to support the model; **build the model**: interpret in mathematical terms the features and relations you have chosen, define variables, equations, draw shapes, gather data, measure objects, etc.; **evaluate and revise your model**: see if your mathematical work makes sense, Atest drive@ the model by examining real and imagined cases to see if it fits the situation or a set of criteria you specified satisfactorily. If it does, use the model until new information suggests it be changed. If not, reconsider the assumptions and revise the model. (Different parts of the above process are studied, first, at different times.) As an additional outcome of this process of criticism, students will understand the difference between universal statements and existential statements. They will understand the advantage of multiple representations of ideas. That is, that some representations demonstrate some aspects of a situation better than others do. For instance, students will see the power of algebra as a way of coding information in some situations. Additionally, they will understand that it often helps to look at several representations of a situation. They also will understand that modeling depends on the quality of the measurements collected. Furthermore, they will understand that function models often are valuable over a limited domain. They will understand the value of Asimplicity@ in modeling. They will recognize the value in checking that what they are doing makes sense *both* mathematically and contextually. They will further recognize that making assumptions includes assumptions about the irrelevancy of some information. Students will focus on models that Achange only one thing at a time.@ Furthermore, they will understand the value of hand tracing and some calculation in models that depend to some degree on technology.

Course 2

Algebra/number/function: Students will deepen their understanding of algebraic inequalities, solving simple ones. They will understand and be able to use absolute value notation (especially as it relates to distance on the numberline). They will be able to solve simple absolute value inequalities such as *x-3*<5. They will understand how the control numbers, **a, h,** and **k** effect the graphs of functions of the form f(x) = **a***x-**h*** + **k**. They will be able to translate functions expressed as sums of such functions (where **a** = 1 and **k** = 0) into piecewise linear representations. (They will understand the concept of a Asum@ of functions.) Students will encounter geometric, growth, and motion modeling problems that lead to quadratic function models. Students also will have considerable experience modeling several types of growth situations using recurrence relations and closed-from algebraic expressions (involving response and explanatory variables). They will continue to deepen their understanding of additive relations ($p_{n+1}=p_n + d$) and multiplicative relations ($p_{n+1}=bp_n$) and their corresponding closed-from expressions ($y=dx +c$ and $y=cb^x$, respectively). They will understand and be able to recognize mixed growth models ($p_{n+1}=bp_n +d$) and the corresponding closed-form expression. They will be able to associate patterns and characteristics which are revealed by plots such as p_n versus p_{n-1}; $p_n- p_{n-1}$ versus p_{n-1}; $(p_n- p_{n-1})/p_{n-1}$ versus p_n; etc. for each of these sequences They will also study sequences of partial sums, such as adding the terms of an arithmetic sequence and adding the terms of a geometric sequence. They also will study the corresponding closed-form expressions for these sequences of partial sums. The closed form is a quadratic in the first case and a (vertically) translated exponential function in the second. Students will also understand and be able to use many other representations in modeling problems such as web diagrams, time-series graphs (p_n versus n), and sign-post graphs. They also will be able to characterize and utilize tabular representations obtained from first differences, second differences, and ratios of successive terms related to linear, quadratic, and exponential models, respectively. They will be able to use characteristics of such models to help choose appropriate models for situations. They will study effects of the control numbers (such as b and d in the recursive expressions above). They will study the concept of *equilibrium* (where $p_{n+1}=p_n$ for all n) and be able to find equilibria in some cases. In addition, for some specific models, they will discuss stability of an equilibrium. Students are introduced to a logarithm as an inverse function for an exponential function, viewing the logarithm as a tool for solving exponential equations. They will focus primarily on the common logarithm (base 10). However, they will discuss how logarithms with other bases are related to the common logarithm. (Through exercises, students will develop common properties of logarithms, although these properties are not stressed in future content.) Students will model several types of motion with algebraic equations and graphs. For instance, they will know and be able to explain why a linear function is a good model of a distance versus time graph for an object moving in a straight line with constant velocity. They will be able to explain the control parameters in such a linear model in terms of velocity and distance. Similarly, they will be able to model an object with a non-constant velocity, but with constant acceleration (such as a falling object or an object thrown upward under the influence of gravity, ignoring air resistance) using a quadratic equation. They will be able to explain the control parameters in terms of the motion of such an object. They will understand the meaning of negative velocity. They will be able to find an function which models such motion (i.e. a quadratic) using at least two method: solving a system of equations resulting from knowing three points on the quadratic curve and using technology to produce a quadratic regression model. Students will have an understanding of average velocity of an object over a time interval and they will have an intuitive understanding of instantaneous velocity. They will be able to connect the concepts of first differences and second differences to average velocity and average acceleration. They will also be able to connect the concept of instantaneous velocity with the geometric concept of the Aslope of a curve at a point.@ They will have at least two methods to find the slope of the curve at a point: they will be able to use technology to zoom in until the curve looks straight, determine two points on the curve, and determine the slope of the line through these two points (giving an approximation); and they will be able to use symmetric difference quotients with linear and quadratic models of motion to find the instantaneous velocity at a specific point and at a general point (Symmetric differences give the exact instantaneous value for these types of functions). They will have experience examining the effect of air resistance by comparing quadratic models with piecewise quadratic models on falling objects. In addition, students will have experience using parametric equations to break the motion of an object in a plane into its vertical and horizontal components. They will have experience relating these equations to the graph of the position of the object in the plane.

Geometry/trigonometry: Students will have significant experience working with a new metric (fire truck distance, also known as taxi cab or Manhattan distance) on the set of integer lattice points in the coordinate plane. They will be able to explain the Ashape@ of circles and ellipses using this metric. They will also know that there is not a unique shortest path between two points in geometry with this metric. They will have experience solving problems related to distance in this new geometry and its one-dimensional analog. They will develop and be able to explain and compare different algorithms that determine a point or set of points that minimizes certain distance combinations. One of these algorithms will require minimizing a set of maximal distances. That is, the algorithm involves finding a *minimax* solution. Students will deductively develop several elementary plane geometry results including the Pythagorean Theorem. They will deductively study other theorems including: the sum of angle measures in a triangle is 180^0; vertical angles are equal; and the length of the side opposite the 30^0 angle in a 30^0-60^0-90^0 right triangle is half the length of the hypotenuse. They will have modeling experience applying geometric results and line symmetry to circle-packing problems such as minimizing the amount of packaging material (perimeter) utilized to package a given number of circles (concentrating on circles of the same size) or maximizing the ratio of area covered by the circles to the area of the Acontainer.@ The concept of Aaspect ratio@ is also encountered. Students will have experience modeling proximity problems using Voronoi Diagrams (i.e. dividing up polygonal regions, each containing Acenters of influence@ into regions determined by proximity to the centers). They will be able to apply geometric knowledge in determining the boundaries of Voronio regions, which consist of line segments on the perpendicular bisector of the segments joining two centers. The will have a variety of ways to construct such diagrams including straight edge and compass, reflecting plexiglas mirrors, paper folding, coordinate geometry (with systems of two equations to determine the Voronoi vertices) and software drawing utilities. They will be able to determine the area of Voronoi regions using triangulation, Pick=s Theorem (unproved) Heron=s formula (unproved) and simulation. Conversely, they will have two strategies to assist in determining the center of a domain divided into Voronoi regions when given the boundaries of the regions. They will understand and be able to use the (coordinate) distance formula in the plane, the midpoint formula for two points in the coordinate plane, and the relationship between the slope of the segment between two points and the slope of the perpendicular bisector of the segment. They will have some experience writing programs to do some of the steps in Voronoi diagram construction.

Probability/statistics: Students will have experience tying statistical measures (the median of a set and the midrange (range between the minimum and the maximum of a data set) to problems in geometry. They will be able to create probability distributions and find the *expected payoff* to the row player in two-person games involving mixed strategies.

Discrete mathematics: Students will have considerable experience modeling certain real world decision-making situations using *game theory* models. That is, they will represent these situations by organizing information in *payoff matrices*. They will be able to classify these models as one of four types: *zero-sum* (or *constant-sum*) *strictly determined* games; *zero-sum* (or *constant-sum*) *not strictly determined*; *non-zero-sum* (*non-constant sum*) with a *dilemm*a; and *non-zero-sum* (*non-constant-sum*) without *dilemma*. (They will also compare some of these game theoretic representations with other representations using tree diagrams and weighted vertex-edge graphs studied in Course 1.) Students will explore these four game model types through actual play and through computer (calculator) simulation using different strategies. They will investigate patterned strategies and random strategies in an effort to optimize their payoff. They will eventually conjecture (based on data collected from play) and justify (using symbolic analysis) a procedure for finding optimal strategies for zero-sum two person games. The procedure requires solving a system containing two linear equations. In the process they will study game *equilibria*, the *value of the game,* and the concept of *fair* games. (Note that while students will have experience trying to optimize strategies in non-zero-sum games with a dilemma, such investigation is still a subject of current mathematical research.) Students will have considerable experience using vertex-edge graphs to model real-life situations involving some type of Aconnection@ between objects and discuss the advisability or value of using such representations in some instances. These situations will include compatibility, conflict, traveling salesperson (TPS), and matching situations. They will have experience converting situations from some tables to vertex-edge graphs and vice versa. Students will have considerable experience developing, writing, and critiquing algorithms relating to optimization problems in Graph theory. They will discuss properties of a Agood@ algorithm (e.g. easy to understand and implement, efficient, and always produces an optimal solution). In addition to creating their own algorithms, students will encounter existing algorithms such as Kruskal=s Algorithm and Prim=s Algorithm for finding minimal spanning trees; the Nearest Neighbor and Brute Force (list all possible routes and pick a shortest one) algorithms related to the TSP; and the Gale-Shapley Algorithm for stable matching. They will also work on unsolved problems related to vertex coloring and the TSP problem. Students will be familiar with graph theoretic vocabulary such as digraph, weighted graph, bipartite graph, connected graph, tree, spanning tree, cycle, Hamiltonian cycle, complete graph, degree, and subgraph.

Mathematical reasoning/modeling/logic: Students will deepen their understanding of mathematical modeling (See Course 1). They will especially have deeper experience formulating criteria, finding solutions to simpler situations and then extending to more general situations while defending and justifying their work. They will also have experience developing more than one model (based on different criteria resulting from different interpretations) for a given situation. They will also understand that modeling is a cyclic process, where, in some cases, when one compares results from a model with the real situation, the initial model is rejected. Students will deepen their ability to choose representations (e.g. tables, graphs and pictures, and symbolic representations) that provide information and insight at various stages in the modeling process. They will deepen their understanding concerning which questions in a context are mathematical and which are not. They will have experience developing Acounterexamples@ to claims that some algorithms in graph theory produce optimal solutions. Students will use both inductive reasoning (to formulate conjectures) and deductive reasoning concerning geometric results. They will discuss the imprecision caused by hand measurement and electronic drawing utilities and understand that even small errors can become significant. They will discuss the value of deductive proof (such as it eliminates imprecision caused by physical measurement) and the Amethod of generalization@ (basically, solving a general problem by utilizing variables rather than a specific problem involving actual numerical measurements.) Students will understand what an iterative process is. They probably also will encounter and understand what a programming Aloop@ is.

Course 3:

Algebra/number/function: Students will develop algebraic formulas and expressions for many modeling situations. For example, they will develop algebraic formulas related and procedures for imaging three-dimensional objects on a two-dimensional plane. As a second example, they will develop several (symbolic) functions to model monetary flow (revenue, cost, profit, profit margin) related to the operation of a small manufacturing business. In this type of modeling, students will strengthen their ability to use linear, quadratic, and piecewise-defined functions. They will also create and analyze (in terms of optimization, for instance) rational functions (with asymptotes) and step functions. In addition, they will create and utilize arithmetic combinations of such functions. Students also will create some two and three variable, real valued functions in modeling situations, like $f(x,y,z) = 32.19x + 38726.5y + 25000\mathrm{int}(x/2000)$, where the expression int(_) refers to the greatest integer function. They then will apply the strategy of holding all but one variable constant and analyzing what a change in the remaining variable does in the model and in the context. (C.f. Course one, where the models created were only one dimensional.) Students will be introduced to the term hyperbola as it relates geometrically to a conic section and symbolically to function of the form $f(x) = a/x$. Students will justify the geometric mean-arithmetic mean inequality and be able to use it to minimize certain functions. (The GM-AM inequality that says that GM #AM). They will strengthen their ability to use closed form formulas for arithmetic sequences. They will create and use expressions such as $P_A/\sqrt{(s(s+1))}$. They will solve rational expressions such as $A/x = B$ for x. They also will create and utilize, and solve simple algebraic inequalities (some involving quotients).

Geometry/trigonometry: Students will be able to represent the image of three-dimensional objects on a two-dimensional plane using mathematical concepts related to five perspective drawing elements: overlapping, diminution, convergence, single and multiple vanishing points, and foreshortening. They will enhance their understanding and ability to work with properties of similar figures and associated concepts of ratios, proportions and scaling. They will also know that two triangles are similar if and only if the three angles of one of the triangles can be put into correspondence with the three angles of the other triangle in such a way that the measures of the corresponding angles are equal. Students will also know, understand why, and be able to use some geometric properties of parallelograms, rectangles, and trapezoids. For example, they will know that the diagonals of a rectangle intersect at a point equidistant from opposite sides. They will be able to use and describe the "convergence" of perspective representations of some three-dimensional parallel lines in a two-dimensional plane. They will begin the study of trigonometry based on properties of right triangles by encountering the sine, cosine and tangent functions as ratios of lengths of sides of a right triangle. Angles are measured in degrees in these contexts. They will pay particular attention to trigonometry related to 30^0-60^0-90^0 and 45^0-45^0-90^0 right triangles. They will know some simple trigonometric identities such as $\sin x = \cos(90^0 - x)$ and $\tan x = \sin x / \cos x$ ($0<x<90^0$). Students will model phenomena that oscillate. They will generate verbal, graphical, and symbolic descriptions/characterizations of oscillating data. They will understand and be able to use the concept of *periodic* data. They will understand the "circular" characterizations/definitions of the sine and cosine functions. They will understand how these circular characterizations relate to and extend definitions of the sine and cosine based on right triangles. They have experience using these functions (and arithmetic combinations of them) to characterize periodic phenomena. They will understand the concepts of *amplitude* and *phase shift*. In fact, they will be able to describe verbally and graphically what each of the parameters (or "control numbers") A,B,C, and D does in expressions like $f(x) = A\sin(B(x-D)) + C$. They will also be able to describe, create, and interpret functions in modeling situations comprised of the sum of a linear and a sinusoidal function; fitting some such combinations to data that has a periodic component and a linear trend. They will have experience graphically "adding" some oscillating functions with different periods and amplitudes together and describing what happens. They will understand *radian measure* and be able to convert from radian measure to degrees and vice versa. They will know some trigonometric identities related to radian measure such as $\cos(t) = \sin(t + B/2)$. Using symmetric differences, they will also understand the cos(t) as a measure of the instantaneous rate of change (intuitively defined) of the sin(t). They will study *damped oscillation*, *damping factors* and *envelopes*. They will be able to create symbolic representations modeling some damped oscillating situations.

Probability/statistics: Students will discuss many issues and develop tools for steps in the entire statistical sampling process: writing a questionnaire, selecting a sample design, administering the questionnaire, compiling the data, analyzing the results, and writing a report. (Moreover, they will have experience with the process from start to finish.) They will deepen their ability to analyze data in tables and graphs. For example, students will be able to use cross-tabulation as an analysis tool. They will understand how wording and information within a question and question ordering can manipulate responses on a questionnaire and introduce one form a bias. They will understand the concept of statistical sampling (and bias issues related to sampling techniques). They will have experience with a variety of sampling techniques (including simple random samples (SRS's), systematic samples, cluster samples, stratified samples, self-selecting samples, and convenience samples); seeing what happens when such sampling techniques are applied to a known population. They will understand (statistical) variability due to sampling. They will have experience determining a sample design, which includes identifying the target population, creating a frame (complete listing), and selecting a sampling technique. They will have in depth experience examining simple random samples from populations in order to detect and measure a specific trait (i.e., infer the proportion of a "yes-no" characteristic) in a population. They will understand the concepts of "point estimates" (e.g. a sample percentage as a point estimate of the population percentage) and "interval estimates" (e.g. confidence interval) for the percentage of a population with a particular trait. They will use simulation and other means to acquire "reference distributions" of likely sample compositions of fixed sample size and a fixed population composition. They will be introduced to the descriptive terms *unimodal*, *multimodal*, and *skewed* as they refer to the shape of histograms. They will understand and be able to apply the concept of a *confidence interval* as a means of establishing upper and lower bounds on the percent of a population with a specific trait (concentrating mostly on 90% and 95% confidence intervals). They will understand the effect of sample size on the size of confidence intervals. They will understand and be able to use the concept of *margin of error*. They will also have a "rule of thumb" for margin of error for a 95% confidence interval and sample size $n \geq 100$; namely $100\%/\sqrt{(n)}$. (They will have a more general rule provided that the sample proportion and sample size together meet certain conditions.) Students will understand the term *statistical significance*. For example, they will be able to tell if two point estimates are *significantly different* by examining the 95% confidence intervals around each estimate. They will examine who sample size effects statistical significance and investigate some sensitivity related to this measure. They also will be able to utilize several of the statistical concepts mentioned above with the Peterson Capture-Recapture method (and, possibly, some of this method's variants) to estimate the (unknown) size of a population.

Discrete mathematics: Students will create and examine existing algorithms related to fair apportionment in a variety of (discrete) situations including the apportionment of identical objects and apportionment of non-identical objects. In the process they will examine the concept of fairness including the development of various criteria upon which to judge fairness (e.g. lack of bias and lack of quota violation) and some quantitative measures of fairness (such as an index of unfairness). They will analyze, compare, and contrast several House of Representatives seat apportionment algorithms, including those of Hamilton, Jefferson, Webster, Adams, and Hill. In the process, they will examine several "rounding" rules that distinguish these methods.

Students will be able to use recursive time-series models to describe some types of interaction between variables. A predator-prey models serves as an example of a two-variable situation. They study "compartment models," e.g., an S-I-R model of the spread of a disease as the compartments of a population ("Suseptibles, Infecteds," and "Recovered") interact with each other. The former constitutes a three-variable system. In a "simpler" one-variable example of a population interacting with itself ("feedback"), students study the recursive logistic model of constrained population growth. In some modeling situations students will develop the recursive formulas for the model. In more complicated, multi-variable cases, students will be given the recursive equations and then analyze the resulting system. For instance, in the single variable case and using a data table, students will know and be able to use a graph of the (absolute) rate of population change (e.g. P(n+1)-P(n) versus n or versus P(n)) or the relative rate of change ((P(n+1)-P(n))/P(n) versus P(n)) to recognize and develop recursive formulas for linear, exponential, and logistic growth. Furthermore, they will be able to interpret the parameters (control numbers) in recursive expressions (e.g. the parameters k and M in the expression $P(n+1) – P(n) = kP(n)(M-P(n))$). They will also be able to use *causal-loop diagrams*, (CLD's), to assist them in determining interactions between components of a system. Students will understand the concept of *joint proportionality* in recursive systems. They will understand the concepts of *carrying capacity* and *intrinsic growth rate* with regard to logistic growth and they will have methods to help them determine these quantities in some contexts. They will also understand concept of a system *equilibrium* and be able to determine one or more equilibria of a in some instances. In two-dimensional systems, they also will understand and be able to use the concept of a *phase-plane* in the analysis of the behavior of the system.

Mathematical reasoning/modeling/logic: Students will enhance their abilities to create and effectively use multiple mathematical representations (graphical, tabular, symbolic and verbal) of real world situations, understanding the value of each representation in the particular context. They will continue to utilize the model process described in this section under the Course 1 heading above. This will include cyclically revisiting the four phases of this modeling paradigm as students more from a simplified situation to a more complex one. Some of the functions students create will occur as a result fitting curves to data: looking at graphical patterns, selecting types of functions to use, fitting these types to the data using technology and by hand, and examining residual plots—and in one case, second differences. However, then these functions are analyzed (and possibly refined) from the point of view of what the function say about the context or the modeling assumptions. In addition, such analysis helps students select among competing function models. Students also have experience weighing competing algorithms (e.g. voting apportionment algorithms) against constraints (e.g. minimizing unfairness). Students will enhance their reasoning skills in many situations, for instance, as they analyze paradoxes related to fair apportionment algorithms. In several situations throughout the year, students examine "sensitivity" of some model parts. E.g. "how well does the model fit if we eliminate the x^2 term with the small coefficients?" or "how many more people would have to have said 'yes' in order for the estimated difference to be statistically significant?" In a summary unit, students select a situation to be modeled, create a model, and write a descriptive and analytical report as well as possibly make a presentation. The unit also contains opportunity to criticize/analyze other modeling reports using assessment criteria that the students create (and refine) themselves and which deal with the internal (mathematical validity and correctness, sensitivity, etc.) and external (alignment of conclusions with reality, etc.) accuracy of the model. Students will experience the power of algebra as a way to eliminate the need for multiple-guess techniques to find a solution.