Integrated Mathematics: A Modeling Approach
Using Technology
(The Systemic Initiative for Montana Mathematics and Science--SIMMS;
The Montana Council of Teachers of Mathematics--MCTM)

Level: 9-12

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Review Materials: Levels 1,2, 3 and 4 were reviewed in first-edition format. Levels 5 and 6 were reviewed in field test version. Complete Teacher editions have been developed and were reviewed and will be available for each Level. Also available is the booklet: Restructuring Mathematics Assessment: Suggestions from the Classroom, and SIMMS/MCTM Integrated Mathematics: A Modeling Approach Using Technology Levels 1-6 Objectives and Content Outline.

Format/Description: This is a complete secondary school mathematics curriculum for all students. It is an integrated curriculum. Mathematical topics are motivated by and presented within applied contexts. A total of six *Levels* of curriculum are available. Each Level comprises a year of mathematics. There are several possible paths through the curriculum. All students are expected to complete Levels 1 and 2, but most students will not complete all six Levels. For example, most students will choose between Levels 3 and 4 (although first taking Level 3 and then taking Level 4 is an option) and most students will choose between Levels 5 and 6 (although first taking Level 5 and then taking Level 6 is an option). Differences between Levels 3 and 4 and between Levels 5 and 6 are apparent in terms of contexts included in the levels. Overlap exists in terms of content in several Levels, particularly between Levels 3 and 4 and between Levels 5 and 6. See the content descriptions below. Some of this overlap could be both review and extension of material. For example, some material on modeling motion using parametric equations in Level 6 is review (depending on a student's path through the curriculum) and extension of material presented in Level 4. Levels 4 and 6 are aligned more with students planning mathematics or science oriented careers. Levels 3 and 5 focus on contexts relating to an informed citizenry. One typical path would be the sequence Levels 1, 2, 4, 6. Such a path might typically be followed by a prospective mathematics or science major. Another possible path would be Levels 1, 2, 3, 5. Either path incorporates solid mathematical development. A third option might be to begin Level 1 in eighth grade and follow Level 6 with an AP course. For more information on these paths and other paths, see the Teacher Edition or the booklet SIMMS/MCTM Integrated Mathematics: A Modeling Approach Using Technology; Levels 1-6 Objectives and Content Outline. Also, see the content descriptions below in terms of content overlap for various paths. Instructional materials at each Level consist of three *volumes* and each volume contains between three and six *modules*. On average, each year contains fifteen modules. Some of these must be presented in sequence; for others, the order is more arbitrary. Prerequisites are listed at the beginning of each module in the Teacher's Edition. Each module begins with a brief overview and introduction to a mathematical topic and contextual situation for at least some of its development. Other contexts or applications are presented later in the module. The module contains several *explorations* followed by *discussions.* Many of these explorations and discussions result in open-ended response. Explorations and discussions are part of a coordinated *Activity.* There are between three and five such Activities in each module. In addition to containing one or more exploration-discussion pairs, each Activity contains *mathematical notes* which include information about the mathematical topics in the activity and an *assignment* to be done outside of class. Some of the mathematical notes may be review for some students depending on their path through the curriculum. Most modules also contain *research projects* at some points during the development. Many modules contain physical experiments or constructions. In addition, each student module contains a *module summary* of the major terms (mathematical terms, or contextual terms from other disciplines as necessary) and concepts from the module, and a *summary assessment*. See the **Assessment** section below.

The Teacher Editions include instructional notes and sample answers for each page of the student materials. Each module in the Teacher Edition begins with a brief overview of its contents followed by a list of teaching objectives and a list of prerequisite knowledge. Time lines are included for each module, as well. Materials, from coins to cardboard, are listed for each Activity. Similarly, technology needs are listed for each module. Some graphing calculator programs which can be used with the materials are included where appropriate. The assignment problems at the end of each Activity in the student materials are listed and separated in the Teacher Edition into two sections by a series of asterisks. The problems in the first section focus on essential elements of the Activity. The second set provides optional additional practice. In addition, some of these problems are designated as suggested assessment items. See the **Assessment** section below. Finally, problem sets entitled "Flashbacks" are included for review of prerequisite skills and are to be used according to the teacher's judgment. Blackline masters and templates are also included.

Pedagogy: The curriculum delivery focus is on student learning. The student materials are written is a way that stresses active student participation. The materials lend themselves to the use of cooperative learning and general class discussions. Such pedagogical techniques are encouraged. In addition, student writing is promoted. The booklet Restructuring Mathematics Assessment: Suggestions from the Classroom contains suggestions with regard to setting clear student expectations, using groups, having students present, using student journals, and documenting student progress.

Technology: Technology is essential for the delivery of this curriculum. In particular, spreadsheets, a graphing utility, a geometric drawing utility, a symbolic manipulator, a statistics package, and the functions of a scientific calculator are used often at each Level. Capability for word-processing and interfaces with some scientific equipment, such as a CBL, are recommended. (The TI-92 provides all of these functions with the exception of spreadsheet applications. Moreover, in some cases the list capabilities of the TI-92 would suffice for spreadsheet applications.) Technology needs for each module are listed in the Teacher Edition at the beginning of the module material. Additional materials such as graph paper or a globe are listed similarly.

Assessment: The use of alternative assessment is encouraged. Summary Assessments in the student

materials provide open-ended questions. Module assessments in the Teacher Editions provide more traditional assessments. In addition, as mentioned above, specific problems in the *assignment* section of an Activity are denoted in the Teacher Edition as potential assessment problems. The booklet Restructuring Mathematics Assessment: Suggestions from the Classroom
lists principles and goals of student assessments. It also contains descriptions of techniques that can be used which provide sources of data regarding a student's ability to communicate mathematically, understand mathematical tasks in context, logically approach a problem, select strategies, form hypotheses and conjectures, select appropriate technology, evaluate the reasonableness of a solution, extend a problem or look for alternate solutions, and maintain a positive attitude toward mathematics. It contains suggestions for assessing group work and individual student progress in a group setting. It discusses self-evaluation, peer evaluation, the use of student interviews, the use of journal writing and presentations and the use of portfolios for assessment purposes. Finally, there is a section on the documentation of student progress including the use of scoring rubrics.

Content Overview: The first four NCTM Standards: Mathematics as Problem Solving, Mathematics as Communication, Mathematics as Reasoning, and Mathematical Connections are addressed in every volume. Students are asked questions throughout the materials. Many open-ended questions are included. They are asked to explain their conclusions or responses often. Connections are made between mathematical topics and between mathematics and other disciplines, particularly scientific disciplines. In addition, subsequent Activities often refer to previous Activities and relate past material to present investigations. The development of the mathematical content is done largely in contextual settings. The contexts are carefully chosen to maintain student interest and demonstrate the relevance of mathematics in real and sometimes recreational settings. Most applications are not contrived and are taken from real settings. Students become acquainted with mathematical terms and results in the course of solving problems.

Our description of what students completing each Level of this curriculum will know and be able to do is given in terms of five "strands": **algebra/number/functions, geometry/trigonometry, statistics/probability, discrete mathematics**, and **logic/reasoning.** As stated above, this is an integrated curriculum in which these strands are woven together tightly. The strands mentioned above are used for descriptive purposes only. Indeed, some mathematical topics could be listed under more than one strand. For instance, vector addition could be listed under **algebra/number/functions** or under **geometry**. However, the decision has been made to list most topics only under one strand in each Level. This decision is based on the development of the topic in the instructional materials with attention given to what additional mathematical topics surround the development. In addition to the latter considerations, it is sometimes extremely injurious to pull topics apart. Moreover, the description below does not necessarily discuss topics in the order in which they are found in the curriculum but rather in an order that makes current reading easier. For the most part, topics that are briefly mentioned but not emphasized, such as the area of an ellipse, are omitted for this description. Furthermore, most concepts from science or some other field, such as finance, that appear in the materials are not mentioned in the descriptions below. It should be noted that these additional concepts will be known to some degree by the student completing a path through the curriculum. In this curriculum, technology is used a great deal for simulations; in addition, it is used to help students recognize patterns, provide evidence for mathematical theorems, or provide calculation capability that might otherwise distract students form major mathematical points. In sum, it is utilized in ways which facilitate student understanding of concepts under study. It does not replace understanding with senseless application of a technological procedure.

Level 1

Algebra/functions/numbers: Given two points, students will be able to compute the slope of a line and recognize that this ratio is constant for any two points. They will know that two parallel lines have equal slopes. They will be able to interpret slope as a rate of change; in addition, they will be able to interpret the slope and y-intercept in an equation representing a real-world situation. Students will identify the domain and range for a linear relation. Given a linear equation, they will be able to write it in slope-intercept form and sketch its graph. They will develop and use the point-slope form for an equation of a line, and be able to convert that form to the slope-intercept form, using the distributive property. Employing appropriate technology, they will graph equations of the form "y=", including linear, exponential, and power (of the form $y=ax^b$) equations. Students will graph two linear equations on the same set of axes and estimate the coordinates of the point of intersection, if there is one. They will also be able to solve systems of two linear equations algebraically, using the method of substitution. They will be able to graph linear inequalities in one or two variables, as well as systems of two or more linear inequalities. They will write a set of linear inequalities to describe a shaded region on a coordinate plane. Students will be able to write and apply an exponential equation of the form $y=ab^x$; in particular, they will develop the generalized growth relationship $T=p(1+r)^n$. They will be able to find the growth rate for a population, and they will explore the effects of the growth rate and the initial population on population growth. Given the growth rate, they will determine the expected value of a population at a specific time. Students will investigate power equations of the form $y=ax^b$ and use them to model real-world situations. They will describe how the values a and b affect the graph of $y=ax^b$. Given a set of data, students will establish whether or not the relationship describes a direct proportion or an inverse proportion. They will write equations that represent a direct or inverse proportion, identify the constant of proportionality, graph these equations on a graphing utility, and compare them. Students will be able to write ratios of corresponding lengths of similar figures (scale factors) and use them to create a proportion to estimate other lengths. They will also be able to solve proportions involving areas or volumes of similar figures. Students will investigate number patterns that form arithmetic or geometric sequences. They will be able to create a spreadsheet to generate a sequence, and they will write a recursive formula to describe the sequence. They will also utilize the spreadsheet to compute an arithmetic series. In addition, they will write and use an explicit formula for calculating the n^{th} term in an arithmetic sequence [$a_n=a_1+d(n-1)$, where d is the common difference between any two consecutive terms] or a geometric sequence [$g_n=g_1r^{n-1}$, where r is the common ratio between any two consecutive terms]. They will compare graphs and equations of arithmetic or geometric sequences with their linear or exponential models.

Students will be introduced to the concept of functions as they investigate step functions. They will be able to graph and interpret a step function, write an equation that models the graph, and explain why the graph represents a function. They will be able to evaluate expressions involving the greatest integer function. They will graph variations of the greatest integer function and observe the effect of changes in a, b, and c on y=aƒx„, y=ƒbx„, and y=ƒx„+d, respectively.

Students will be able to compute the percent of increase or decrease in a quantity; in particular they will explore trends in used car trade-in values by looking at amounts of depreciation as well as the percent decrease and comparing these two methods for representing loss in car value. They will convert metric and U.S. standard units of measure (e.g. cm to dm; cm^3 to L; ft^3 to yd^3). They will convert some rates to different units, such as L/sec to L/hr.

Geometry: By examining regular polygons formed by reflections in two hinged mirrors, students will discover and apply several relationships between the number of sides and angle measures (e.g. the sum of the measures of the interior angles of a regular polygon is 180E(n-2)). They will explore paths that light rays travel in a reflection and compare angles of incidence and reflection. They will make many conjectures about relationships involving reflection (e.g. a line of reflection is the perpendicular bisector of the line segment connecting an image point to its preimage point; the image of point (x,y) is (x,-y) under a reflection in the x-axis). They will be able to sketch the image of a figure reflected in the x- or y-axis. They will use properties of reflection to solve real-world problems such as finding the shortest path between three locations and determining the path of a bank shot in billiards. Students will know the sum of the measures of the exterior angles of any regular polygon. They will be able to find the measure of an exterior or interior angle of a regular polygon and use this information to determine which regular polygons tessellate a plane. Moreover, they will explore tessellations involving non-regular polygons as well. With the aid of a geometry utility and a spreadsheet, they will construct regular polygons and develop a formula for finding area. Students will investigate the concept of similarity by examining the relationship between scale factor and the areas or volumes of two similar figures. They will discover and apply their conclusions that the ratio of the areas is the square of the ratio of the lengths of corresponding sides and the ratio of the volumes is the cube of the ratio of the edge lengths.

Students will be able to identify objects that are prisms and explore the number of bases, lateral faces, and lateral edges. They will create nets for prisms and be able to find the surface area of a prism. They will estimate the area of an irregular region by following a step-by-step procedure (counting unit squares) or finding the mean radius and computing the area of a circle that approximates the region. They will be able to use this result to estimate the volume of a solid with the irregular region as its base. They will be able to compute the volume of a right prism and a cylinder. Students will build a model of a 3-D coordinate system. They will be able to plot ordered triples and identify coordinates of points in a 3-D system. Given its coordinates, they will sketch a figure in 3-D. They will also use a matrix of coordinates and a symbolic manipulator (such as Mathcad or Mathematica) to create a surface plot, i.e. a 3-D graph representing the surface of an object.

Probability/statistics: Using simulations of simple lotteries (some involving a random number generator), students will be introduced to concepts in probability. They will be able to describe a sample space for various situations, determine and compare experimental and theoretical probabilities of events, and realize that the sum of the probabilities for all possible outcomes of an experiment is 1. With the aid of a spreadsheet, they will explore and discover a pattern that will generate ${}_nC_r$ (not stated as such) and use the results to compute probabilities. They will be able to compute the expected value for a game such as a simple lottery and ascertain whether or not the game is fair. Students will use Venn diagrams (2- and 3-circle) to organize data and help determine probabilities of events. They will create tree diagrams to organize information, show all possible outcomes for an experiment, and help in probability calculations. They will apply the fundamental counting principle to find the total number of outcomes of an experiment.

Students will collect data for a variety of situations and be able to use a spreadsheet to organize and analyze data. They will be able to create single and double bar graphs and interpret information from them. They will distinguish between subjective and objective information. Given a table of data for two quantities, students will employ technology (spreadsheet, graphing utility) to construct a scatterplot and use it to describe a relationship between the quantities as a positive association, negative association, or neither. They will be able to draw a linear, exponential, or power (i.e. of the form $y=ax^b$) model for a set of data, write an equation for the model, and use it to make predictions or estimates. They will explore the sum of absolute values of residuals (difference between observed and predicted values) as a method to determine accuracy of a linear, exponential, or power model. They will also calculate percent errors as an alternative method for finding a line of best fit. Students will use simulations to model real-world situations such as population growth, oil spills, and spread of a disease. They will compare and contrast data generated from a simulation with expected data generated by a spreadsheet.

Discrete Mathematics: Students will be introduced to some fundamental ideas in graph theory as they explore the problem of visiting several rooms in a school in the shortest amount of time. They will distinguish between a path, circuit, and Hamiltonian circuit. They will draw weighted graphs and investigate the brute force algorithm, nearest neighbor algorithm, and cheapest link algorithm as strategies to determine optimal routes. Students will use tree diagrams and/or the fundamental counting principle to find the total number of different routes between vertices. They will become familiar with factorial notation and develop the formula (n-1)! for finding the number of possible Hamiltonian circuits in a complete graph. Given a linear programming problem, students will write a linear objective equation and a set of constraints (linear inequalities), graph the feasible region and identify the corner points, and apply the "corner principle" to determine the coordinates of the point that maximizes the objective equation.

Logic and reasoning: Students will formulate conjectures based on explorations and experimentation and write or discuss arguments that support these conjectures. They will be introduced to the idea of mathematical theorems as "conjectures based on patterns [that] are stated in a general form and proved."

<u>Level 2</u>

Algebra/functions/numbers: Students will be able to solve proportions involving similar triangles. As they investigate various methods of apportionment as a means of fairly representing a population, students will evaluate and/or simplify some complex algebraic fractions and compute the geometric mean $\sqrt{ab}$ of two numbers a and b. They will explore solving systems of linear equations, within the context of linear programming problems (see Discrete Mathematics below), by reviewing the substitution method and using matrix equations. Students will investigate exponential growth and decay, and they will write equations of the form $y=ab^x$ that model given situations. Given an equation of the form $y=ab^x$, they will identify the initial quantity a and growth or decay rate r (where b=1+r), and they will have strategies (such as graphing the function and using a guess-and-check method, but not using logarithms) to solve for y, b, and x. They will graph exponential equations for both positive and negative exponents using appropriate technology, interpret these representations, and use them to make predictions.

Students will examine properties of exponents ($b^x \cdot b^y=b^{x+y}$; $(b^x)^y=b^{xy}$; $b^x/b^y=b^{x-y}$) by means of a spreadsheet to generate examples. They will convert expressions (algebraic or numerical) containing negative exponents to expressions with positive exponents, and vice versa.

Geometry/trigonometry: Students will determine whether or not two triangles are similar, using the "AAA Property." They will use the Pythagorean Theorem to find the length of a side of a right triangle and to develop and apply a formula for the distance between two points. They will use the converse of the Pythagorean Theorem to ascertain whether given lengths of sides will form an acute triangle, an obtuse triangle, or a right triangle. Students will compute areas of regular polygons and circles. They will explore and compare surface areas and volumes of various 3-D figures; in particular, they will calculate total surface area and volume for right prisms, cylinders, pyramids, and cones. They will investigate the relationship between volumes of cylinders and cones with congruent bases and between volumes of right prisms and pyramids with congruent polygonal bases and the same height. They will use formulas to find the volume and surface area of a sphere. Students will be introduced to an intuitive idea of limits by exploring how an increase in the number of sides of a regular polygon inscribed in or circumscribing a circle results in areas that approach the area of the circle. They also examine the idea of a cylinder or a cone as "the limit of a progression" of regular polygonal right prisms or pyramids whose bases are inscribed in the same circular base. Students will extend their knowledge of similar figures as they explore transformations in the Cartesian plane. They will graph an image under a dilation centered at the origin, a translation, a rotation about the origin, or a reflection (through the x- or y-axis, the line y=x, or the line y=-x). They will be able to determine the scale factor for a dilation. They will write matrix equations to describe dilations and translations. They will use matrix multiplication to produce an image under a rotation about the origin or a reflection. They will examine rotational symmetry in regular polygons and lines of symmetry in several figures. They will realize that the line of reflection is the perpendicular bisector of the segment connecting a point in the preimage with its image. They will perform a composite transformation on a figure (e.g. a reflection followed by a dilation). They will recognize when a transformation is an isometry. Using paper folding and a geometry utility, students will explore geometric properties of quadrilaterals (rhombus, parallelogram, rectangle, square, trapezoid) and circles. They will formulate conjectures about relationships (e.g. the diagonals of a rhombus are perpendicular bisectors of each other) based on their observations. They will know that vertical angles are congruent and that corresponding angles, alternate interior angles, and alternate exterior angles formed by two parallel lines cut by a transversal are congruent. They will be able to find the center of a circle (or center of rotation) by constructing perpendicular bisectors of at least two chords.

Students will explore the tangent, sine, and cosine ratios for similar triangles and recognize that each is constant for a given acute angle. They will apply these ratios to find the length of a side of a right triangle. They will use the $\tan^{-1}$, $\sin^{-1}$, and $\cos^{-1}$ keys on a calculator to determine angle measures in a right triangle. Using the appropriate trigonometric ratios, they will prove that $\tan\angle A=\sin\angle A/\cos\angle A$. They will develop and test some conjectures regarding trigonometric relationships, such as $\sin\angle A=\cos\angle B$ for acute angles A and B in a right triangle.

Probability/statistics: After collecting and organizing data, often using simulations (e.g. randomly drawing a ball from a container to model a fishing derby at a carnival), students will be able to calculate experimental probabilities. They will be able to compute the probability of the complement of an event and recognize that the sum of the probabilities of two complementary events is 1. They will be able to find theoretical probabilities of events by constructing Punnett squares, drawing tree diagrams, and creating geometric models. They will compare experimental and theoretical probabilities. Students will explore probabilities of two or more events; in particular, they will use the formula $P(A \text{ and } B) = P(A)\cdot P(B)$ for two (or more) independent events and the formula $P(A \text{ or } B) = P(A)+P(B)-P(A \text{ and } B)$ for any two events, including mutually exclusive events. They will find probabilities for multistage experiments involving either independent or dependent events (conditional probability is defined in a Mathematics Note). They will compute the expected value of a game and determine whether or not it is a fair game.

Students will investigate various data representations; in particular, they will create and interpret frequency tables, histograms, "stacked" histograms, pie charts, stem-and-leaf plots, back-to-back stem-and-leaf plots, and box-and-whisker plots. They will utilize these visual displays to compare sets of data, and they will consider the advantages and disadvantages of each representation. In addition, they will explore the use of "accurate statistics to support an inaccurate or deceptive position." Students will be able to compute the mean, median, mode, and range for a data set, and determine which measure of central tendency best represents the given data. They will find the upper and lower quartiles, the interquartile range, and any outliers for a set of data. They will calculate the mean absolute deviation for a set of data and consider the effects of a change in one or more data values. They will compute the standard deviation for a set of data, using a formula or a calculator, and observe where the data values occur within one, two, or three standard deviations. Students will be introduced to a variety of sampling methods that attempt to identify particular characteristics of a population, including simple random sampling, stratified sampling, and systematic sampling. They will perceive possible sources of bias in sampling techniques and survey questions. They will employ a simulation to generate samples, and they will create a histogram of sampling data that can be used to estimate probabilities. Students will graph the standard deviation equation $\sigma=\sqrt{p(1-p)/n}$ and determine how sample size (n) affects the maximum standard deviation. They will be familiar with confidence statements, and they will use the formula $2\sigma=1/\sqrt{n}$ to calculate the margin of error and make predictions about a given characteristic of a population. Students will investigate methods for creating linear models for sets of data and use them to make predictions. They will compute the sum of the absolute values of the residuals to evaluate and compare linear models. They will find the median-median line and decide whether it appears to model given data. They will determine the regression line for a set of data using the least squares method. They will create and analyze a residual plot to determine whether or not a linear model is appropriate for a set of data.

Discrete Mathematics: Students will be introduced to the idea of matrices as a means for organizing and analyzing information. They will be able to perform matrix addition and subtraction as well as scalar multiplication. They will evaluate the product of two matrices, if possible, and interpret its results. They will explain why matrix multiplication is not commutative. They will solve a system of two linear equations by writing the system as a matrix equation and utilizing technology to find the inverse of the coefficient matrix. They will discover that this process fails for an inconsistent system, since the coefficient matrix does not have an inverse in this case. Students will solve linear programming problems by writing and graphing the set of constraints (linear inequalities), identifying the feasible region, finding the corner points, writing the objective function, and using the “corner principle” to determine the maximum or minimum value of the objective function. [An optional activity extends linear programming to three variables.] Students will investigate sequences and be able to generate terms in an arithmetic or geometric sequence from an explicit formula. Given a number sequence, they will explain whether it is arithmetic, geometric, or neither, and write an explicit (closed form) or recursive formula, if one exists. They will develop and apply a formula for the sum of the terms of a finite arithmetic sequence. They will follow an algorithm to develop and use a formula for the sum of the terms of a finite geometric sequence. They will also examine the formula for an infinite geometric series in which the common ratio is between -1 and 1. They will explore the behavior of a sequence as the number of terms increases indefinitely; moreover, they will utilize technology (spreadsheet, graphing utility, geometry utility) to determine whether or not values in a sequence appear to approach a limit.

Logic and reasoning: Students will enhance their ability to formulate conjectures and develop arguments to support their reasoning (see **Geometry/trigonometry** above). They will investigate several methods for writing algorithms as means for delineating a set of clear, specific instructions to complete a task. In particular, they will consider characteristics of an efficient algorithm and write a set of step-by-step instructions to perform tasks such as determining a prime number or programming a robot to trace a simple path. They will create flowcharts to model algorithms such as the Euclidean algorithm for finding the greatest common divisor of two natural numbers. They will explore recursion in algorithms as it pertains to generating sequences (see **Discrete Mathematics** above) and geometric figures. For example, they will design an algorithm that recreates a given diagram of inscribed pentagons.

Level 3

Algebra/functions/numbers: Students will explore situations that can be modeled by polynomial equations, primarily linear and quadratic (cubic equations are introduced), and they will use these equations to make predictions. In particular, they will create and analyze distance-time graphs. They will compute and distinguish between the average speed and average velocity of an object over a particular time interval. They will estimate instantaneous velocity. They will calculate average acceleration and relate the coefficients of a quadratic equation to the acceleration, initial velocity, and initial position of an object [$d(t)=-½\, gt^2+v_0t+d_0$]. Students will investigate the effects of the values a and b on power equations of the form $y=ax^b$ by looking at their graphs.

Geometry/trigonometry: Students will draw three-dimensional coordinate systems, plot points using ordered triples, and develop the formula for finding the distance between two points in 3-D. They will apply these ideas as they explore, interpret, and create topographic maps that describe geographic features; for example, they will find the distance between locations of a microwave transmitter and its receiver on a topographic map in order to determine appropriate sites for repeaters that would enable microwaves to actually be transmitted to the receiver. Students will investigate the classical solids (Platonic solids and Archimedean solids) in terms of the number and type of regular polygons that form the faces, the sum of the measures of the interior angles at each vertex, and planes of symmetry. They will explore the relationship among the faces, vertices, and edges (Euler's formula). They will sketch nets for some solids (e.g. cube, regular tetrahedron). They will be able to determine whether or not a polyhedron is a Platonic solid. They will calculate the surface area of regular polyhedra given edge length. They will find dual or reciprocal polyhedra for the Platonic solids and examine the relationship between the number of faces and the number of vertices. Students will explore relationships among lines of symmetry and sides of regular polygons (e.g. lines of symmetry that intersect sides of regular polygons are perpendicular bisectors of those sides). Students will model construction of the conic sections (ellipse, circle, hyperbola, parabola) and explore their geometric properties using a geometry utility and paper-folding. They will investigate the reflective properties of curved surfaces (in particular, parabolas) and explain their use in real-world applications such as satellite dishes and car headlights. Students will discover the golden section N in ratios of dimensions of objects such as golden rectangles, golden triangles, and some human measurements (e.g. the length of an arm from shoulder to fingertips : the length of the same arm from elbow to fingertips) and explore arithmetic relationships involving N (e.g. N=1+1/N). They will analyze various geometric representations and proofs of the Pythagorean Theorem.

Students will review right triangle trigonometric ratios and the Pythagorean Theorem by calculating lengths of sides of right triangles. Using a geometry utility, they will explore the relationships sin2=sin(180-2) and cos2=-cos(180-2). They will develop and apply the Law of Sines to find lengths of sides or measures of angles of non-right triangles. They will compute areas of triangles using trigonometric ratios, i.e. Area=1/2 absinC. Students will investigate the relationship between a^2+b^2 and c^2 for acute, right, and obtuse triangles as they are introduced to the Law of Cosines, which they use to determine lengths of sides or measures of angles of non-right triangles.

Probability/statistics: Students will create tree diagrams and be able to apply the Fundamental Counting Principle to determine all possible outcomes for a given situation. They will explore permutations and combinations, including the relationship between them. They will be introduced to notation and formulas for ${}_nP_r$ and ${}_nC_r$. They will compute permutations and combinations and use these counting techniques to calculate probabilities. Students will be introduced to binomial experiments as they investigate procedures such as those involved in quality control for a manufacturer, where they will recognize that taking a small random sample without replacement from a large population can be modeled by a binomial experiment, since changes in probabilities are relatively small. They will develop the binomial formula for computing the probability of r successes in n trials: ${}_nC_rp^rq^{n-r}$, and apply it to finding theoretical probabilities. They will create a table and a histogram to represent the probability distribution for a binomial experiment. They will calculate the expected value for a binomial experiment and determine the expected region for a given percentage of the total probabilities.

Students will be able to create scatterplots of data and observe any possible trends in the data. They will investigate various types of equations that can be used to model data. In particular, they will apply the principle of least squares to determine a power equation of the form $y=ax^b$ that can model a set of data. They will also utilize technology to find linear, exponential, and power regression equations for data sets and determine which equation best represents the data by comparing their graphs or examining the sums of the squares of the residuals. They will explore situations in which an exponential or polynomial regression equation may not be a suitable model for making predictions beyond the given data (e.g. cyclic nature of sunspot activity).

Discrete Mathematics: Students will enhance their ability to solve linear programming problems by writing a system of linear inequalities in two variables to describe the constraints on a given situation, graphing the feasible region, finding the corner points, writing the objective function that describes a quantity to be maximized or minimized, and using the “corner principle” to determine the maximum or minimum value of the objective function. In order to determine the corner points of the feasible region, they will review solving systems of two linear equations by graphing and algebraically, using the method of substitution. In addition, they will solve a system of two linear equations by writing the system as a matrix equation and utilizing technology to find the inverse of the coefficient matrix. [An optional activity extends linear programming to three variables where students explore solving three-variable systems of equations using substitution and matrices.] Students will explore and apply strategies for fair division of continuous or discrete items among two or more people. For example, they will use the “reduction” method to simulate the division of a candy bar among three people and the “bid-and-divide” method to determine the final settlement values for two people who inherit an antique car. Students will review and expand upon the concepts of graph theory that were introduced in Level 1. They will draw graphs to represent locations (vertices) and routes between them (edges) and determine whether or not a graph is complete, connected, traversable, has a Hamiltonian circuit, or contains a closed Eulerian path. They will be able to establish the degree of each vertex in a graph. They will develop and apply relationships between types of graphs and their vertices: e.g. a traversable graph contains either all vertices of even degree or exactly two vertices of odd degree; a closed Eulerian path contains all vertices of even degree; “A digraph has a closed Eulerian path if and only if each vertex has the same in-degree and out-degree.” Given a digraph, students will create a matrix for the number of one-, two-, or three-edge paths between two locations and list all possible paths. They will also draw a digraph for a corresponding matrix of one-edge paths and use digraphs to model and analyze optimal route situations (e.g. finding an efficient snowplow route). Students will explore Fibonacci (and Fibonacci-type) sequences, generate some of these sequences, and observe that the limit of the ratio of consecutive terms is the golden section N. In addition, they will follow a procedure to generate Pythagorean triples from four consecutive terms of Fibonacci sequences. Students will investigate simple and compound interest using spreadsheets and scatterplots. They will apply formulas involving simple interest ($I=Prt$ and $A_t=P(1+rt)$). They will develop and apply formulas involving compound interest ($A_t=P(1+r)^t$ and $A_t=P(1+r/n)^{nt}$). They will observe that an account balance appears to approach a limit as the number of compounding periods increases. They will determine the annual percentage yield for a given interest rate and they will estimate the doubling time for an investment. Students will explore amortization by creating a spreadsheet and using "guess-and-check" to determine monthly payments. In addition, they will apply formulas for the monthly payment of an installment loan and for the balance on a loan after a given amount of time.

Logic and reasoning: Students will investigate the truth value of compound statements involving the logical connectives *and* and *or*, negations, and conditional statements. They will create two-circle Venn diagrams containing appropriate subsets of elements to represent compound statements. They will develop truth tables for compound statements involving *and*, *or*, and *not* and for logically equivalent statements such as -(p and q) and -p or -q. Students will determine the truth value for conditional statements, form a truth table for p6q, write the contrapositive of a conditional statement, and apply the principle of contrapositive reasoning to form logical conclusions. They will implement a variety of strategies (tables, diagrams, the process of elimination, contrapositive reasoning) to solve logic problems.

Level 4

Algebra/functions/numbers: Students will know the general form of a polynomial: $y = a_nx^n+a_{n-1}x^{n-1}+...a_1x+a_0$, where n is a non-negative integer. They will know that constant functions are polynomials where n=0. They will know the terms *coefficient, degree, and root* related to polynomials. They will have considerable experience working with polynomials which *split*; (the term split is not used) that is, polynomials that can be written as $y=c(x-a_1)(x-a_2)(x-a_3)...(x-a_n)$. They will be able to use the distributive property to write a polynomial in such a factored form in general polynomial form. (They do not explore the opposite direction, factoring, at this time.) They will understand the connection between the zeros (roots) of a polynomial and linear factors of the polynomial. They will have at least one strategy for "fitting" polynomials to data sets. They will have at least on strategy for estimating the degree of a polynomial from its graph by looking at its zeros and turning points. They will understand that some graphics curves can be modeled splicing together pieces of cubic polynomial curves (Bezier curves). They will be able to predict the "end behavior" and range of some graphs including those with "multiple zeros" such as $y=a(x-b)^n$ by noting the sign of a and the parity of the integer n. They will know the terms *quadratic* function and *parabolic* curve. They will recognize the symmetry of a parabolic curve and know that the vertex of the parabola lies on the curve. They will be able to use quadratic functions in many modeling situations. Students will have explored and know the basic properties of (integer) exponents. They will understand logarithms and be able to use them to solve equations and in other situations where the exponent is the unknown. They will be able to use base 10 log plots and logarithmic scales to display data that covers a wide range of values. They will have explored and know the basic properties of logarithms (with any positive base) including the relation between logarithmic and exponential equations. They will understand the meaning of (finite) intervals of real numbers and be able to use conjoined inequalities to describe such intervals. They will be able to solve such inequalities as : 4# 3x+1# 7 using arithmetic operations with inequalities. Most, but not all, of such inequalities involve linear expressions. Combining these concepts, students will be able to determine a “precision” interval in the domain of a linear function that provides (maps into) a “tolerance” interval in the range of the linear function. They will be able to demonstrate the concept of a precision interval for a tolerance interval for linear functions graphically. Students will connect these latter ideas in the concept of *limit* of a function as x approaches a number *a* in the domain of the function. They will understand the absolute value *x-a* as representing the distance between x and a and translate such expressions as a-b<x<a+b into *x-a*<b and vice versa. Students will understand what rational functions are. They will be able to describe rational functions using domains, discontinuities (here, isolated points at which the function is not defined), and asymptotes (vertical, horizontal, and oblique.) They will be able to analyze end behavior (and whether or not there is a linear asymptote) by using technology (a symbolic manipulator) to represent a rational function as Q(x) + R(x)/d(x), where R(x)/d(x) approaches zero as *x* gets large and Q(x) is a polynomial. They will be able to graph solution sets for systems of (usually one or two) inequalities of the form y# f(x) or y< f(x), where f(x) is a rational function using technology (to graph f(x)). They will be able to model some real-life applications using rational functions. They will understand that two functions are equivalent if and only if they have the same domain and the same value at each point of their domains. Students will be able to categorize functions by family characterized by a “parent” function. That is, they will be able to graphically, verbally and symbolically describe functions that belong to a particular category, such as the category of sine functions described by the parent y=sin x. Usually, functions belong to a family if they are a “transformation” of the parent. A transformation of the parent y=f(x) is most often a function of the form y= af((1/b)(x-c)) + d. Usually only one of the parameters a,b,c, or d is appears at a time. Moreover, students will be able to describe graphically and verbally what a change in one of these symbolic parameter does. Broader families are also considered. In particular, students will have the most familiarity with families of exponentials, logarithms, periodic functions and some rational functions. Students will also be able to model linear and circular motion with parametric equations. In addition, they will have experience establishing appropriate domains (for the parameter) and in determining ranges (in the plane) in some of these modeling situations. They will have some experience converting linear parametric equations to non-parametric form.

Geometry/trigonometry: Students will be able to model many cyclic events with functions. They will understand radian measure for angles and understand its relation to arc length in a circle. They will be able to work with directed angles. They will understand the concept of periodic function and know the circular definitions of the sine and cosine functions. They will recognize the graph of a sine function. Moreover, they will understand how the various parameters in y=A(sin(B(x+C))+D) effect the graph in terms of amplitude, horizontal shift, vertical shift, and period length. They will be able to determine the absolute maximum and minimum values for functions whose graph resembles a sine function. They will understand how the cosine function is a translation (horizontal shift) of the sine function. They will be able to use angle measures in the interval (0,B/2) to find other values of the sine function. Students will review basic geometric plane transformations including reflection, rotation, transformation, glide reflection, and dilation and view these operations as functions. They will understand the terms *image* and *preimage* as they relate to these transformations. They will recognize glide reflection as the composition of a reflection with a translation. They will have experience looking for patterns arising from compositions using reflections. They will have strategies to determine two dimensional square matrices represent a reflection of a plane polygonal figure in a line through the origin or a rotation of the figure about the origin. By "represent," here, we mean students will be able to find a 2x2 matrix to reflect *n* points, represented as 2x*n* matrix, using matrix *multiplication* to perform a given reflection. They will be able to represent translation of points using *sums* of two matrices. They will have experience finding and using homogeneous matrix forms (3x3 matrices) for transformations and points with the operation of matrix multiplication to represent some of basic geometric transformations of the plane using matrix multiplication. They will recognize matrix multiplication as function composition. (All basic transformations can be represented using homogeneous forms.) They will understand the meaning of an *oriented* triangle in the plane and be able to determine whether or not a transformation preserves or reverses orientation. Students will understand the concept of *vector*. They will be able to add vectors graphically to obtain the "resultant" vector. Moreover, given the resultant and one vector, they will be able to find a vector which, when added to the given vector, gives the resultant vector. They will be able to find components of vectors from the origin on a coordinate plane. They will be able to describe such vectors by determining their *component vectors* and by using and/ or finding magnitude and directional angles. They will be able to add vectors algebraically using component vectors. They will understand *scalars* and scalar multiplication of vectors related to the geometry of vectors. Through guided discovery, they will develop justification for the Law of Cosines and the Law of Sines. They will be able to use both of these laws in vector analysis. They will be able to use vector analysis in many contexts including some that involve displacement, velocity, force, and/or bearings. Students will understand and recognize the classical conic sections: circle, ellipse, hyperbola, and parabola. They will be able to describe these shapes in several ways: as they are related to a plane cutting a double cone, as they represent a locus of points in a plane, and as are the solution set of points in the coordinate plane which satisfy certain equations. They will be able to work with the standard rectangular coordinate equations for each conic and will have developed other equations derived from geometric distance properties relative to locus of points definitions of the curves. They will be able to use a symbolic manipulator to relate different equations for the same curve. They will be able to use a geometry drawing utility to construct these curves using various locus of points descriptions. Students will have experience working in spherical non-Euclidean) geometry. They will compare many results in Spherical geometry with their counterparts in Euclidean geometry. They will have collected experimental evidence to support facts such as: two distinct lines intersect in two points in spherical geometry (so, Euclid's Fifth Postulate does not hold), the sum a of the measures of angles of a triangle in spherical geometry exceed 180^0, and similar polygons are congruent in spherical geometry. They will provide some arguments to support such facts as the sum of angles in a quadrilateral exceeds 360^0 in spherical geometry and their exist triangles with two right angles in spherical geometry. They will have investigated the fact that the area of a triangle in spherical geometry is related to the sum of its angle measures minus 180^0. They will have been introduced to Gauss' formula for the area of a triangle in spherical geometry.

Probability and statistics: From given data sets, they will be able to create tables and graphs of frequencies and relative frequencies. In particular, they will be able to create relative frequency histograms and polygons. They will simulate some data sets with coin tossing using different numbers of coins and/or technology. They will understand the difference between discrete and continuous probability with regard to the range of values that an experiment can take on. They will recognize the theoretical probability distribution measuring the number of heads from tossing *n* fair coins as a *binomial* or *Bernoulli* distribution. There is a little exploration with binomial experiments where p … ½ . They will know the theoretical mean and standard of a binomial distribution, at least with p = ½. They will explore both uniform (continuous) and normal distributions. They will know and be able to use the 68%-95%-99.7% rule regarding standard deviation in normal distribution. They will see that their binomial distribution graph approximates the graph of a normal distribution. They will use both binomial (with p = ½) and normal distributions (using technology) to approximate some data sets. Students will be able to compare sample statistics including the sample mean and the sample standard deviation to the corresponding population statistics. They will understand and have developed experimental support for the Law of Large numbers. They will understand and have developed experimental support for the Central Limit Theorem as it relates sample means to the normal distribution. They will know and be able to use the "standard deviation of sample means" for samples of size *n*. They will understand and be able to create confidence intervals; particularly 68%, 95%, and 99.7% confidence intervals.

Discrete Mathematics: Students will understand the concepts of chromatic number of a (standard) map where countries are connected regions. They will have strategies for determining the chromatic number for maps. These strategies include inspection as well as theoretical results. For example, they will know that the chromatic number of a map consisting of regions inside a square which have been determined by a collection of straight lines passing through the region is two. They will understand the concept of dual graph of a map as well as the concepts of network, complete graph, and vertex degree of a graph. They will also be able to convert map coloring problems to a coloring problem within graph theory involving dual graphs and use a greedy algorithm to color maps. In addition, they will model and solve scheduling problems using coloring theory within graph theory. They will determine a relationship between the chromatic number of a complete graph and the number of vertices in the graph. They will investigate the four-color theorem for maps on a sphere or plane. They will have investigated Heawood's equations for maps on closed 2-dimensional surfaces with *g* holes (that is, surfaces of genus *g*, although that term is not used.) They will understand the concept of equivalent graphs and have experience determining when a graph is planar (that is, can be embedded in the plane without intersecting edges). For instance, they will know that K_5 (the complete graph on five vertices) is not planar. They will explore embedding graphs on other surfaces. Through guided discovery, students will develop formulas for permutations, ${}_nP_r$, and combinations, ${}_nC_r$ and they will be able to use these concepts in many counting contexts.

Logic and reasoning: Students will develop their ability to phrase written statements as conditional statements. They will be able to identify *hypotheses* and *conclusions*. They will be able to represent conditionals using Venn diagrams and determine the truth value of conditionals and negations using truth tables. They will understand and be able to write the *converse, inverse,* and *contrapositive* of a conditional statement. They will understand what logically equivalent statements are. They will understand the meaning of *if and only if*. They will understand the meaning of universal and existential quantifiers and be able to rewrite and interpret statements using quantifiers. They will understand the law of syllogism (referred to, here, as "transitivity") and have some experience providing deductive reasoning using chains of conditionals. They will understand the concepts of *direct proof, indirect proof*, and *proof by exhaustion*, as well as the role of *counterexamples.* They will have some experience utilizing each of these concepts to establish logical results. They will have some exposure to the *explore-conjecture-prove* paradigm. They will be introduced to the concepts of *axioms* and *theorems.* (They do not work with axiomatic systems at this time.)

Level 5

Algebra/function/number: See the conic section and vector algebra descriptions in *Geometry/trigonometry* below.

Geometry/trigonometry: Students will explore curves with constant width other than the circle, namely, Rouleaux polygons which are formed using regular polygons with an odd number of sides. They will develop experimental evidence that the width of such polygons between *lines of support* is constant. They will see that analogous constructions using regular polygons with an even number of sides do not produce curves of constant width. They will develop and be able to use a formula for the perimeter of Rouleaux polygons. In addition, they will develop and be able to use a procedure to determine the area enclosed by a Rouleaux polygon. They will explore "real-life" uses or potential uses of these shapes. Students will understand the four conic sections (circle, ellipse, parabola, and hyperbola) in terms of planes cutting a double cone. They will collect experimental physical evidence of the light reflective properties of conics and understand the concept of *focus* as it relates to these reflective properties as well as its role in defining conics as a locus of points. They will know the standard rectangular coordinate forms of equations for an ellipse, a parabola, an hyperbola as well as equations for the asymptotes of a hyperbola and be able to work with and transform some equations involving the conics, sometimes using a symbolic manipulator. They will design conic curves in applications requiring certain reflective characteristics. Students will understand and be able to use the concept of a *vector* to model displacement, velocity and force. They will be able to represent vectors geometrically and as an ordered pair with the first coordinate representing the magnitude and the second coordinate representing the reference angle. They will be able to add vectors geometrically (tip-to tail) and algebraically using right triangle trigonometry. They will know that vector addition is associative and commutative. They will be able to determine the horizontal and vertical components of a vector both geometrically and symbolically as well add vectors using their horizontal and vertical components. Students will understand and be able to use *radian* measure for angles. They will understand the relationship between radian measure and the length of an arc subtended by the angle on a circle of radius *r* centered at the vertex of the angle. The will extend the meaning of the trigonometric (right triangle) ratios to circular functions, where the sine and cosine measure the y- and x-coordinates, respectfully, of the point of intersection of the unit circle and the terminal ray of an angle measured counterclockwise from the x-axis. They will be able to compute the sine and the cosine of some angles, such as 2B/3, geometrically. They will also view the sine and cosine functions as the y- and x-coordinates, respectfully, when a vertical coordinate line tangent to the unit circle at zero is wrapped around the line. They, then, will know that the domain of the sine and the cosine functions is the set of real numbers, when the angles are measured in radians. They will understand the concept of a periodic function and be able to measure the period for some graphs of sine and cosine functions and in some applications such as sound waves. The will understand what a *cycle* is for a periodic function. They will understand the concept of amplitude for a sine and for a cosine functions and be able to determine amplitude given a graph or a symbolic formula such as y=2sinx. Indeed, they will know how changes in the parameters *a* and *b* effect the graph in functions of the form y=asin(bx). The will view the tangent function as the quotient of the sine and cosine functions, when this quotient is defined.

Probability and statistics: Students will be able to represent the probability involving one and two-dimensional visual situations using geometric representations. They will be able to translate these probabilistic geometric models into expressions involving symbolic expressions involving multi-stage experiments. They will be introduced to the Law of Large Numbers and create experimental evidence for its validity through simulation. They will be introduced to the concept of *random variables*. They will have some experience working with discrete random variables the associated probability distributions. They will also know how to determine the expected value of a discrete random variable. They will work with the binomial distribution and have seen and be able to work with formulas for its mean and standard deviation. They will have simulated random sampling using technology and other techniques. They will have discussed differences between sampling with replacement and without replacement. Students will be able to estimate the mean and standard deviation of a population using a (random) sample. They will understand what the sampling distribution of sample means (for samples of a fixed size) is and the relation between the standard deviation of sample means and the standard deviation of the population. Similarly, they will estimate the percent of a population with a certain characteristic (relative frequency) using sampling and recognize that the sampling distribution is similar to the binomial distribution in shape. They will be introduced to the Central Limit Theorem and the shape of the normal curve and have developed experimental evidence for its validity. They will have discussed the role of sample size and know, among other things, the "rule of thumb" that a (random) sample used to estimate population parameters should be of size at least 30. They will understand and be able to use confidence intervals and the 68%-95%-99.7% confidence levels to estimate the population mean. They will have developed experimental evidence for the "68%-95%-99.7% rule" related to one, two and three standard deviations from the sample mean (or relative frequency). They will understand what a *null hypothesis* and *alternate hypothesis* are. They will have some experience with hypothesis testing. They will also have had some experience with "capture-recapture" methods for estimating population size. Students will understand and be able to use several statistical tools in order to evaluate the use of a regression line to model the association of two variables in a data set. Specifically, they will understand and be able to use the concepts of *total variation* and *explained variation* to compute the *coefficient of determination,* r^2. They will also know and be able to use the *linear regression coefficient*. That is, they will be able to use the coefficient of determination and/or the regression coefficient to explain linear association (or lack of linear association). They will be able to interpret the sign of the regression coefficient with regard to the slope of the regression line. They will know and be able to compute the *average deviation of the prediction* and the associated *approximation intervals*. That is, students will know that if # *r*#. 1, then "most of the time" a dependent datum will fall within two average deviations of the value predicted by the regression line.

Discrete mathematics: Students will augment skills using the nearest neighbor and cheapest link algorithms in contexts with weighted graphs. In addition, they will be able to analyze scheduling problems using network diagrams and critical paths. They will have some experience using bin packing to analyze problems. Students will be able to construct, interpret, and use *transition matrices* in Markov chain problems. They will be able to determine n^{th} *state vectors* given an initial state vector using matrix multiplication and powers of the transition matrix. They will understand and be able to recognize a *stable (*or *steady*) *state matrix* and *stable vector* relative to the Markov process. They will understand and be able to identify *regular* Markov processes. Students will understand the term *algorithm* as a step-by-step procedure. They will have had experience writing some specific and some general algorithms. They will know that more than one algorithm will accomplish a given task in some circumstances. They will have thought about characteristics of "good" algorithms. They will be able to use the **scan-conversion** algorithm used for displaying graphs of functions and relations in computer graphics. They will have experience using transformation matrices and symmetry of graphs to increase efficiency of the above algorithm. Students will have investigated properties of two-person zero-sum games. In particular, they will be able to construct and interpret *payoff matrices*. They will know what a *strictly determined* game is. They will be able to look for and interpret *saddle points*, when they exist. They will understand what a *mixed strategy* is and have some experience finding optimal mixed strategies for non-strictly determined games. They will understand that a mixed strategy for a player is optimal when the expected payoff value for that player doesn't change regardless of the strategy of the other player. They will know that an optimal strategy exists and develop experimental evidence for this fact using technology. They will understand the concept of the *value of the game* for both strictly determined and non-strictly determined games and be able to determine that value for many games.

Logic and reasoning: Students will understand the use of the quantifiers *all*, *some*, and *none*. They will understand what it means for a quantified statement to be true, at least in a finite universe of discourse. They will be able to rephrase negations of quantified statements into logically equivalent statements; for example "not all dogs are smart" is equivalent to "some dogs are not smart." They will be able to represent a conditional statement symbolically (p6q), graphically using Venn diagrams, using and everyday language. They will be able to describe to components of a conditional in terms of a hypothesis and a conclusion. They will understand the difference between the inclusive and the exclusive *or*. They will be able to determine the truth value of statements involving *and*, *or*, negation and implication using truth tables. They will know De Morgan's Laws with regard to conjunction and disjunction (these terms are not used). They will have investigated truth-value connections between a conditional statement, its converse, its inverse, and its contrapositive. They will know that a conditional statement and its contrapositive are logically equivalent.

Level 6

Algebra/number/functions: Students will develop both recursive and closed form formulas for compound interest money amounts. They will determine the Euler constant, *e*, as the intuitive limit of the sequence $(1+1/n)^n$ as n approaches infinity and view this limit symbolically as $\lim_{n64}(1+1/n)^n = e$. They will have some additional experience with other limits based on this result. They will be able to use the continuous interest formula $A = Pe^{rt}$ in finance problems. They will understand the concept of natural logarithm, lnx, and be able to use this concept to solve problems in the context of finance problems such as the amount of time it takes to double an investment when there is compound or continuous interest. Students will be able to identify several characteristics of polynomial and rational functions. For example, they will know that a polynomial of degree *n* can have at most a total of *n*-1 "peaks and valleys." For polynomials of degree *n*, the end behavior of the function is the same at both "ends" (+4 and -4), if *n* is even and the behavior is "opposite" if *n* is odd. Moreover, a polynomial of even degree will have an odd total number of peaks and valleys and a polynomial of odd degree will have an even total number of peaks and valleys. Given a symbolic expression for a polynomial function (and at least some rational functions), f(x), students will be able to modify the expression to reflect various transformations of the graph of the original function. For example, f(-x) corresponds to the reflection of the graph in the y-axis, -f(x) corresponds to a reflection in the x-axis, f(x)+k corresponds to a vertical translation, f(x+k) corresponds to a horizontal translation, kf(x) corresponds to a vertical "stretch or shrink" and f((1/k)x) corresponds to a horizontal stretch and shrink. Using a symbolic manipulator, students will be able to write a rational function, f(x), as the sum of a polynomial and a rational function, p(x)+r(x)/q(x), where the degree of q(x) is greater than the degree of r(x). They will know that f(x) approaches p(x) asymptotically. They will have considerable experience examining the case where $r(x)/q(x) = k/(x+c)^n$. Using this knowledge, students will be able to create symbolic expressions of functions that have certain prescribed graphical behavior. Students will understand modular arithmetic. They will understand what it means for two integers to be congruent mod *n* to mean the two numbers have the same remainder when divided by the positive integer *n*. They will also understand congruence mod *n* geometrically by wrapping a number line around a circle which has circumference *n*. The will be able to add and multiply in modular arithmetic. They will understand and use the concepts of additive identity, multiplicative identity, additive inverse, and multiplicative inverse (when an inverse exists). They will use the above concepts to solve linear equations in modular arithmetic. Students will be able to represent functions using set diagrams (also called arrow diagrams) and mapping diagrams. They will be able to find domains and ranges for simple polynomial (mostly linear), exponential, and trigonometric functions (sine, cosine and tangent) and their inverses, when they exist (over restricted domains for the trigonometric functions.) They will be able to find *implicit domains* when functions are only given in terms of formulas and *contextual domains* when functions refer to contexts. (These terms are not used.) They will enhance their ability to find domains and ranges when two functions are added, subtracted, multiplied or divided, or composed. They will explore the composition of two functions algebraically. They will know, for example, that composition of functions is not commutative. They will understand what it means for a function to be *one-to-one*. They will be able to determine whether or not a function is one-to-one using the graph of the function and the horizontal line test. They will be able to graph the *inverse* of a one-to-one function given the graph of the original function. They will be able to find the inverse of a one-to one function algebraically for simple functions, such as linear functions. They will understand when a relation is a function. They will be able to use standard notation, f^{-1}, for the inverse of a function ,*f*. They will understand that the inverse of an exponential function is a logarithm function. They will understand the concept of a *relation* and the *inverse* of a relation. Students will develop and be able to work with parametric equations for modeling motion; particularly for a projectile whose path is a parabola, a circle, or an ellipse. They will be able to use vector analysis and trigonometry to determine some parametric relationships. Students will understand *complex numbers* as an extension of the real number system. They will understand the standard representation, a + b*i*, of a complex number, where i^2 = 1 or *i* = %(-1). They will also understand and be able to use several other representations of complex numbers: ordered pairs of real numbers, vectors, and the trigonometric form r(cos2 + *i*sin2). They will understand the terms *modulus* and *argument* as they pertain the trigonometric representation of a complex number. They will be able to perform addition, subtraction, and multiplication of complex numbers using the standard representation. They will be able to find multiplicative *reciprocals* of non-zero complex numbers and use this idea to perform division. They will know that addition and multiplication of complex numbers is commutative and associative. They will be able to represent addition and subtraction using the ordered pair and vector representations. They will understand the value of the trigonometric representation when multiplying complex numbers. They will have seen a statement of De Moivre’s Theorem regarding powers of complex numbers. They will have explored the value of De Moivre’s Theorem as it pertains to the vertices of regular polygons in the plane. They will have seen a statement of the *Fundamental Theorem of Algebra* and have explored experimental evidence for its validity. They will have seen a statement of and worked with the quadratic formula. They will understand its value in actually determining the roots of a quadratic function guaranteed by the Fundamental Theorem of Algebra. They will have seen the value of both the Fundamental Theorem of Algebra and De Moivre’s Theorem in determining the n^{th} *roots* of a complex number, *z*.

Geometry/trigonometry: Students will be able to locate points in the plane using polar coordinates and points in three-space using cylindrical coordinates. They will have explored the geometry of two kinds of projection of portions of the sphere onto the plane or cylinder: stereographic projection and cylindrical projection. They will have explored the relationship between stereographic projection of a point and polar coordinates of its image. In addition, they will have explored the relationship between cylindrical projection and the rectangular coordinates of its image on a plane strip formed by “unwrapping” the cylinder. Students will be able to construct a coordinate lattice for the integers mod *n*. They will describe finite geometries on such a lattice for particular values of *n* be describing lines as the set of points in the lattice that satisfy linear equations. Similarly, concepts such as parallelism and perpendicularity are described and studied algebraically in these finite geometries. Students also will have studied finite geometries axiomatically, particularly the geometries of Fano and Young. Students will model periodic phenomena using trigonometric functions. This modeling includes simple models using one trigonometric function, such as f(x) = 3sin(2x) + 5, to more complicated models using sums of trigonometric functions, such as h(x) = cos(3x) + 5sin(2x). Some of this modeling involves looking at scatterplots. They will be able to find amplitudes and periods of such functions. They will understand the effect of each parameter in an expression
g(x) = af(b(x-h)) + k, where f(x) a trigonometric function. They will explore the other periodic trigonometric functions: tangent, cotangent, secant, and cosecant, particularly by comparing each function to its co-function in terms of domain, range, and graph. They will explore and develop some trigonometric identities, such as sin(-x)= -sin (x) and be able to use them to rewrite some functional expressions. Students will have explored the relationship between the average rate of change of a function and the slope of the tangent line, particularly when the function represents the displacement of

an object. They will have encountered the terms *instantaneous velocity* and *instantaneous rate of change*. They will have encountered the term *derivative of a function at a point* and be able to interpret the derivative at a point as the slope of the curve at that point. They will also have experience computing derivatives algebraically, with quadratic functions, and experience computing derivatives using technology, such as with the function $f(x) = \sqrt{(25 - x^2)}$. They will have an intuitive understanding of the expression $\lim_{h\to 0}(f(t + h)-f(t))/h$ and use this to define the derivative. They also will be able to use the symbol $f'(x)$.

Probability/statistics: Students will enhance their ability to form null and alternative hypotheses. They will understand the difference between "failing to reject the null hypothesis" and "accepting the null hypothesis." They will understand the two errors that can occur in hypothesis testing: rejecting a true null hypothesis and failing to reject a false null hypothesis. They will be able to use z-scores from a table to test hypotheses regarding population parameters for normally distributed populations. They will be able to compare z-scores to the 68%-95%-99.7% rule for normally distributed populations. They will enhance their ability to use the Central Limit Theorem to test null hypotheses about sample means. They will know that a sample size of at least 30 is desirable in this process. In particular, they will understand and be able to compute z-scores for sample means, which leads to rejecting or failing to reject null hypotheses at various significance levels. They will concentrate on the 0.1, 0.05, and 0.01 significance levels. Students will be able to determine conditional probabilities in both theoretical and empirical (experimental) settings. They will be able to use standard notation for conditional probabilities. They will understand the concept of *independence* of two events to mean that the probability of one event happening does not change when it is conditioned upon the other event happening. They will be introduced to the terms *random variable* and *probability distribution of a random variable*. They will know the properties of a *binomial experiment*. They will have developed the formula for *r* successes in a binomial experiment with *n* trials. They will have developed Pascal's Triangle in the context of a game and understand its relationship to combinatorial coefficients. The will have explored some other patterns and uses of Pascal's Triangle, such as its use in determining the coefficients of the terms in the expansion of $(x+y)^5$ (the *Binomial Theorem*) They will have been introduced to the concept of *expected value* of a (finite) random variable and have used this concept in some applications. They will understand the concept of a *fair game* in terms of expected winnings. Students will be able to use the chi-square, χ^2, statistic to test hypotheses at various significance levels regarding difference between expected and observed frequencies and to test whether or not two variables are independent. They will be able to recognize and describe the general shape of a chi-square distribution. They will understand the terms *degrees of freedom* and be able to compute degrees of freedom for two situations: experiments where the trials are independent and the theoretical probabilities for each outcome remain unchanged each time the experiment is performed, experiments used to test the independence of two events. They will be able to compute the chi-square statistic when measuring differences between expected values and observed values. They will be able to use tables to compute the chi-square statistic for various significance levels.

Discrete mathematics: Students will understand how the terms constant, linear, polynomial, exponential, and power describe sequences. They will understand how to use finite differences to determine the lowest possible degree of a polynomial which generates a sequence when given a finite number of terms of the (infinite) polynomial sequence. Using technology, they then will be able to determine a polynomial which generates (at least the given finite number of terms of) the sequence. They will know such results as: there will be a non-negative integer n such that the n^{tk} sequence of differences is constant if and only if the sequence is a polynomial sequence and that a polynomial sequence is generated by a unique polynomial. They will have some experience working with *recursive* sequences and finding "explicit" (*closed form)* formulas for recursive sequences. They will also have some experience generating sequences from situations. In addition, students will have investigated processes with functions and geometric constructions. They will have been introduced to qualitative definitions of *dynamical systems* and *chaos*. They will be able to use the term *orbit* to describe the sequence of values when a real-valued function of a real variable is iterated using an initial value. They will be able to investigate orbits using *web diagrams.* They will be able to determine *fixed points* of such functions both geometrically and algebraically (for linear or quadratic functions). They will be able to classify fixed points as *attractors* or *repellors* in some situations. They will be able to classify orbits as *fixed*, *periodic* (and look for its *cycle*), or *chaotic*. They also will have investigated the *logistic model* of population growth.

Logic/reasoning: Students will have experience proving theorems using the methods of *direct proof, indirect proof,* and *exhaustion* in these finite geometries and in some other contexts. They will understand *contrapositive* reasoning. That is, they will be able to form the contrapositive of conditional statements and they will know that a conditional statement is true exactly when its contrapositive statement is true. Students will know and be able to use the Principle of Mathematical Induction as well as the modified Principle when the base case is greater than one. They will use induction mostly with problems involving formulas for sums or comparisons expressions involving the positive integers as well as some geometric situations. They will have experience making conjectures as to what this base case is. They will also understand that the verification of a few cases does not verify a universal statement. They will also understand that a base case must be proved.