Asymmetric Fractal Trees: Koch Canopies and Dragon Curves

Emma Anderson (Ithaca), Jack Krueger (Concordia), Bianca Teves (Haverford)

Ithaca College Dynamical Systems REU, Summer 2021

Symmetric Fractal Trees

A symmetric fractal tree has a critical scaling ratio r which is dependent on the branching angle θ. This ratio r is the point at which the limits of every branch connect to form a curve. Figure 1 shows two symmetric fractal trees at their critical r-values:
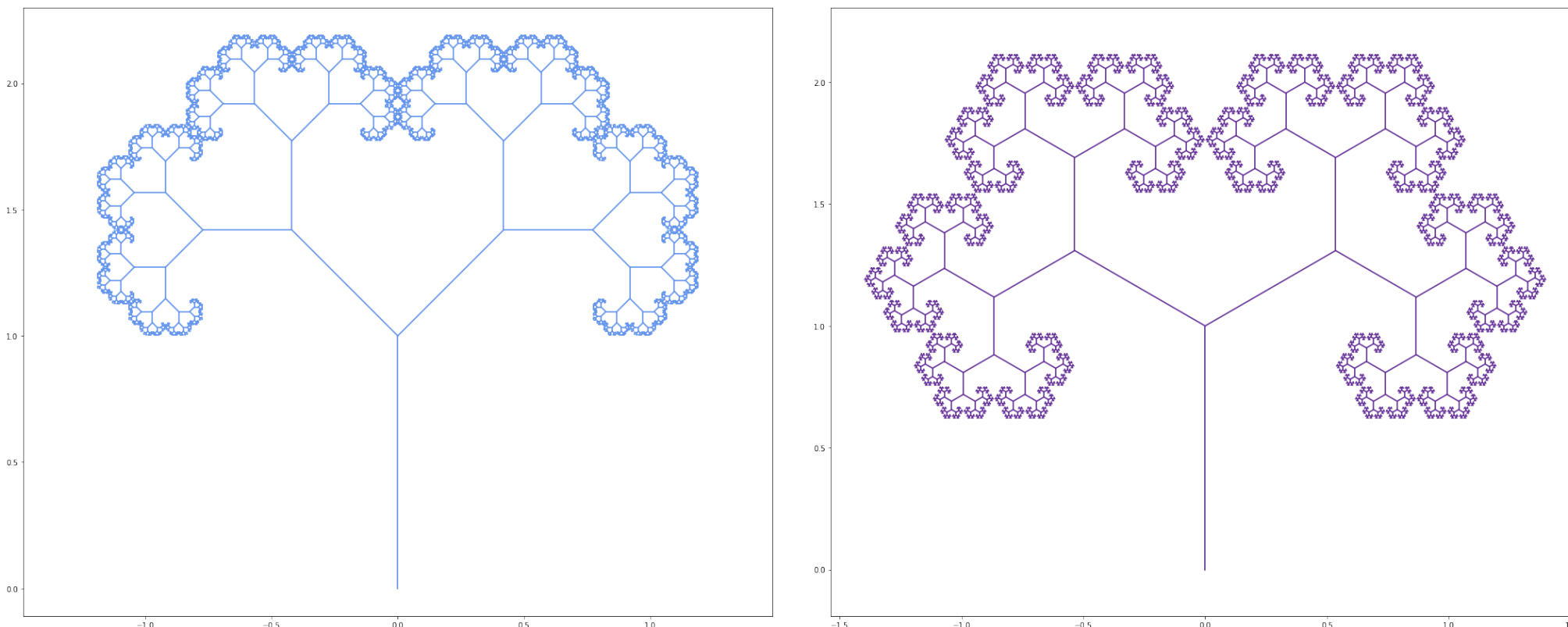

Figure 1:Symmetric Trees: $\theta = \frac{\pi}{4}, \theta = \frac{\pi}{3}$

Figure 2 shows the canopies of the trees from Figure 1, which consist of the tips of each branch in the final iteration of the tree.
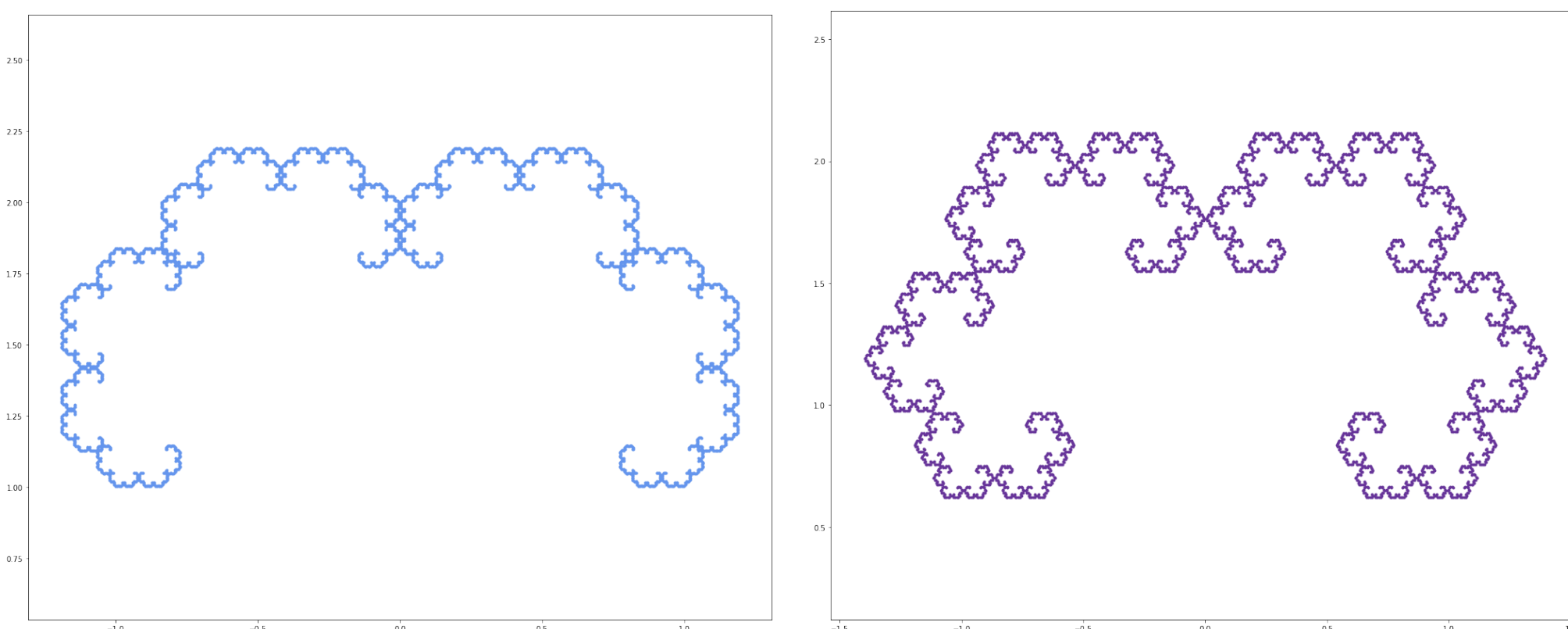

Figure 2:Symmetric Canopies: $\theta = \frac{\pi}{4}, \theta = \frac{\pi}{3}$

Benoit Mandelbrot devised a way to solve for r using the branching angle θ and the coefficient d_n [1], which denotes the number of right turns minus the number of left turns in an address:

$x = r\sin(d_1\theta) + r^2\sin(d_2\theta) + r^3\sin(d_3\theta) + \cdots + r^n\sin(d_n\theta) + \cdots$

$y = 1 + r\cos(d_1\theta) + r^2\cos(d_2\theta) + r^3\cos(d_3\theta) + \cdots + r^n\cos(d_n\theta) + \cdots$

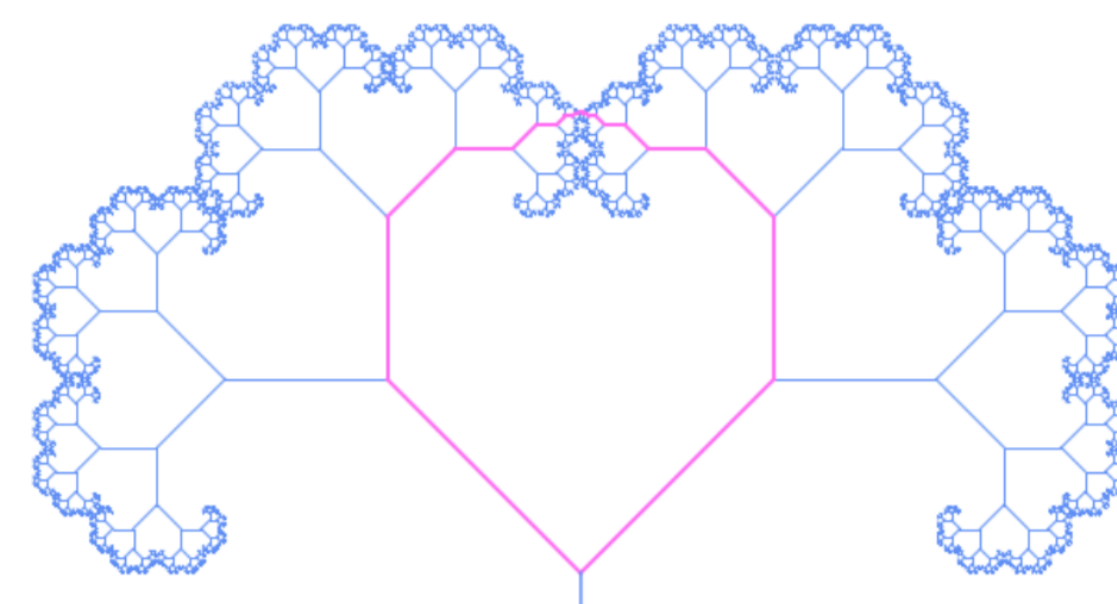Figure 3:Self-contact at $RL^3(RL)^\infty, \theta = \frac{\pi}{4}$

For values of θ such that $0 < \theta \le \frac{\pi}{2}$, one self-contact point occurs at $RL^{N+1}(RL)^\infty$, where $N\theta \ge \frac{\pi}{2}$ [1]. By setting the x-coordinate of this point equal to zero, we can solve for r.

References

[1] Benoit B. Mandelbrot and Michael Frame.
The canopy and shortest path in a self-contacting fractal tree.
The Mathematical Intelligencer, 21(2):18–27, 1999.

[2] Benoit B. Mandelbrot.
The Fractal Geometry of Nature.
W.H. Freeman and Co., 1st edition, 1982.

[3] Larry Riddle.
Koch curve, 1998.

Shifted Trees

One way to implement asymmetry in fractal trees is to shift one branch down by a factor of $(1-s)$, as shown in Figures 4 and 5:
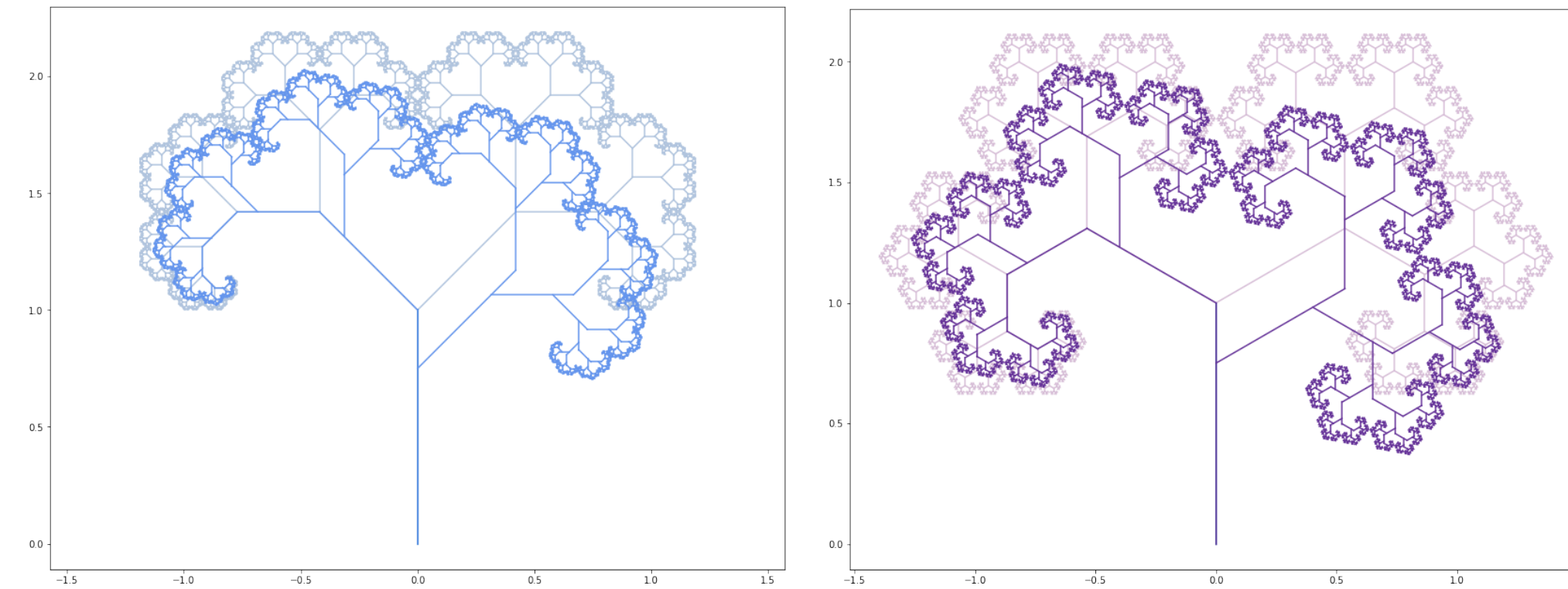

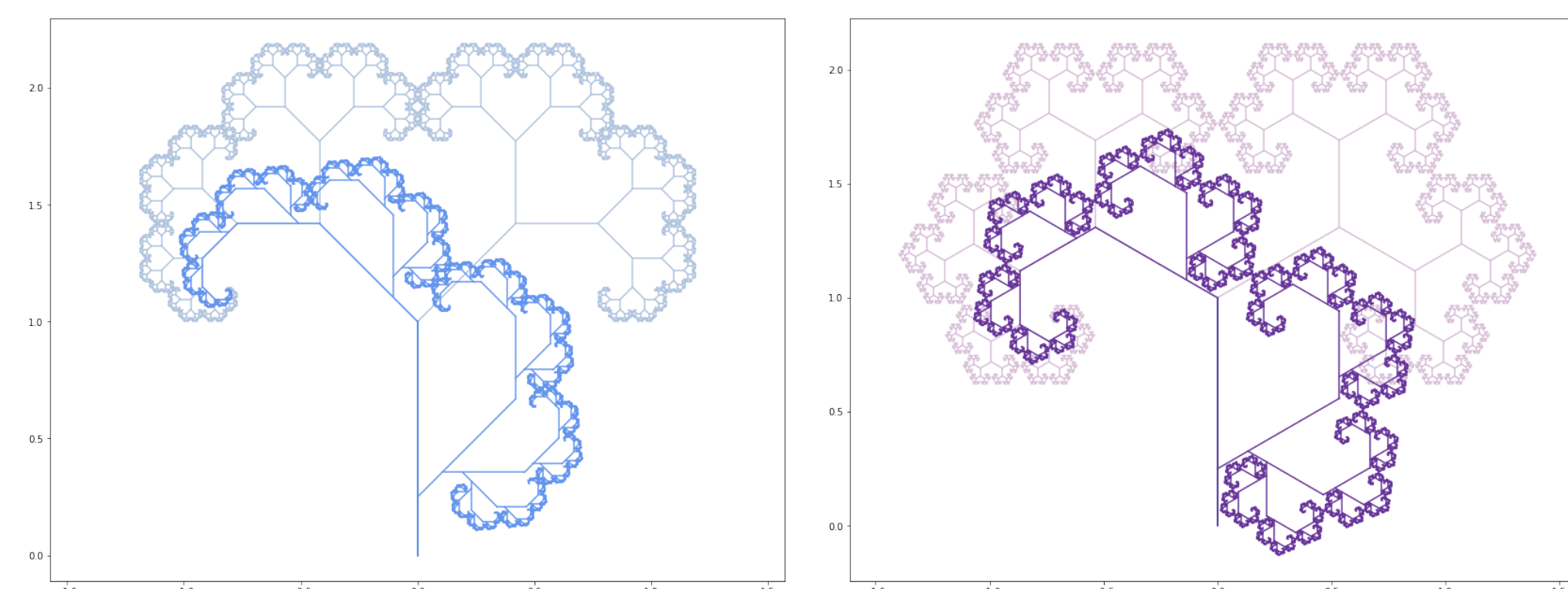Figure 4:Shifted Trees: $\theta = \frac{\pi}{4}, \theta = \frac{\pi}{3}, s = 0.75$

Figure 5:Shifted Canopies: $\theta = \frac{\pi}{4}, \theta = \frac{\pi}{3}, s = 0.25$

We can solve for r by incorporating s into our given equations:

$RL^2L'(RL')^\infty : x = r\sin(\theta) + r^2\sin(0) + r^3\sin(-\theta) + sr^4\sin(-2\theta) + \cdots$

$\qquad y = s + r\cos(\theta) + r^2\cos(0) + r^3\cos(-\theta) + sr^4\cos(-2\theta) + \cdots$

Affine Transformation

$$f_s\begin{pmatrix} x \\ y \end{pmatrix} = D_{L'R \to RL'}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} L'R_x \\ L'R_y \end{pmatrix}$$

Tilt Angle

As seen in Figures 4 and 5, the shifted canopies appear to be tilted at an angle ϕ, which is given below:

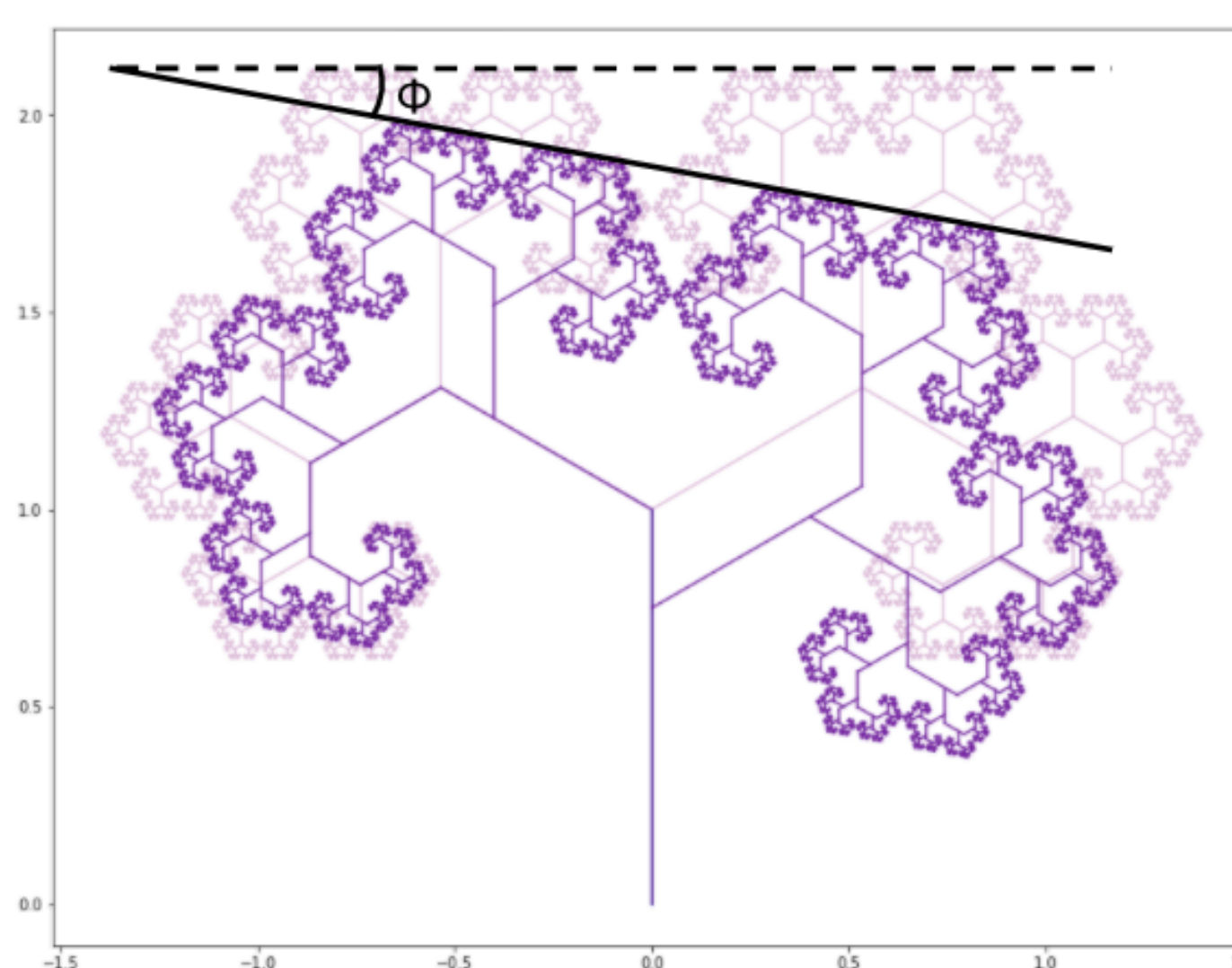$$\phi = \arctan\frac{(1-s)\cdot(1-r\cos\theta)}{(1+s)\cdot r\sin\theta}$$

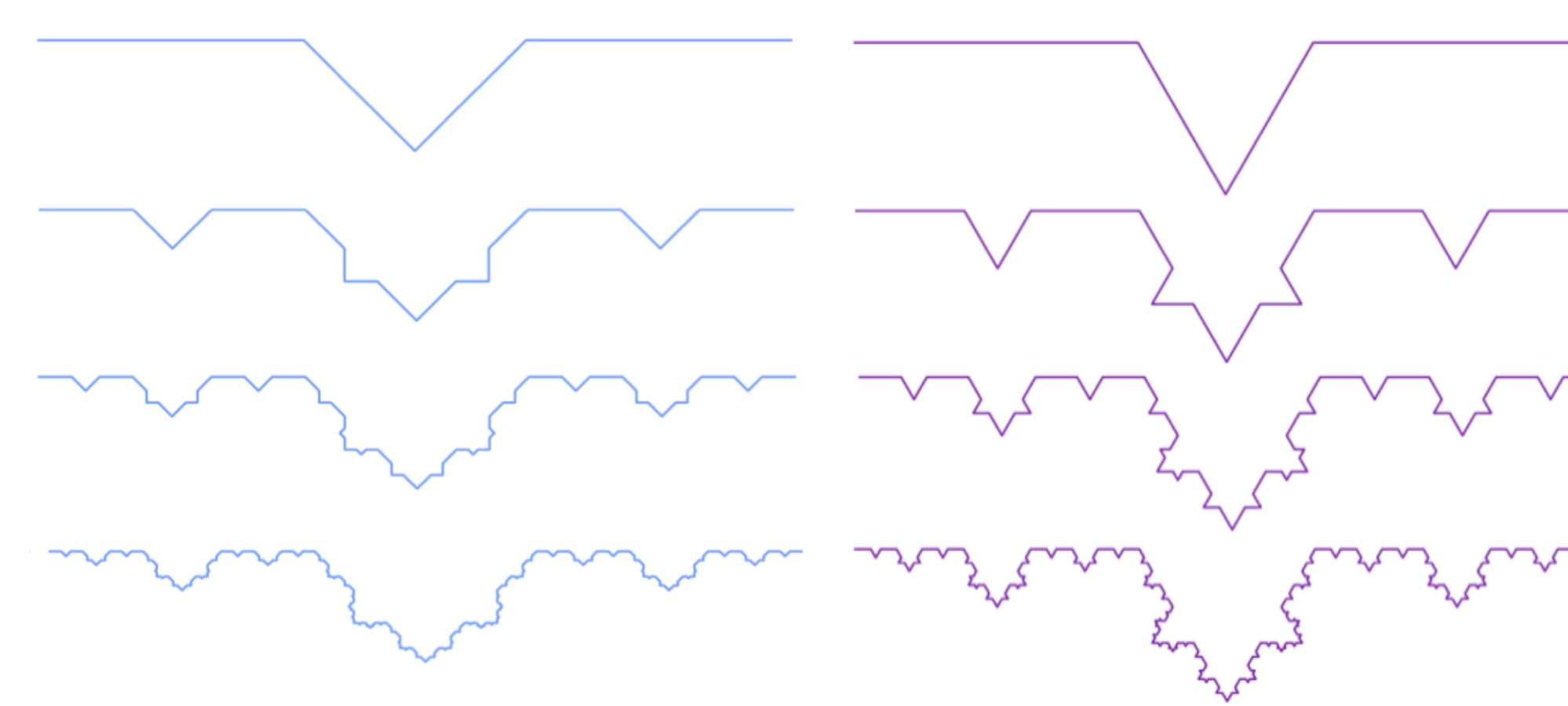Figure 7:Tilt Angle ϕ, $\theta = \frac{\pi}{3}, s = 0.75$

Koch Curves

Figure 6:Four Iterations of Koch Curves: $\theta = \frac{\pi}{4}, \theta = \frac{\pi}{3}$

The Koch Curves shown in Figure 6 are constructed using an Iterated Function System, shown below. This IFS is based off of a similar construction for the Koch Snowflake [3]. q_1 and q_2 are ratios which compare the horizontal and diagonal line segments to the distance between the furthest left and right points.

$$f_1\begin{pmatrix} x \\ y \end{pmatrix} = q_1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$f_2\begin{pmatrix} x \\ y \end{pmatrix} = q_2\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} q_1 \\ 0 \end{pmatrix}$$

$$f_3\begin{pmatrix} x \\ y \end{pmatrix} = q_2\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -q_2\sin\theta \end{pmatrix}$$

$$f_4\begin{pmatrix} x \\ y \end{pmatrix} = q_1\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1-q_1 \\ 0 \end{pmatrix}$$

Mapping the Koch Canopy

We can scale our Koch Curve by r and rotate it by θ, then plot those smaller curves around the first one. This gives us a curve anchored at $(0,0)$. By applying the affine transformation given above, we can map the curve to fit the canopy onto its corresponding fractal tree:
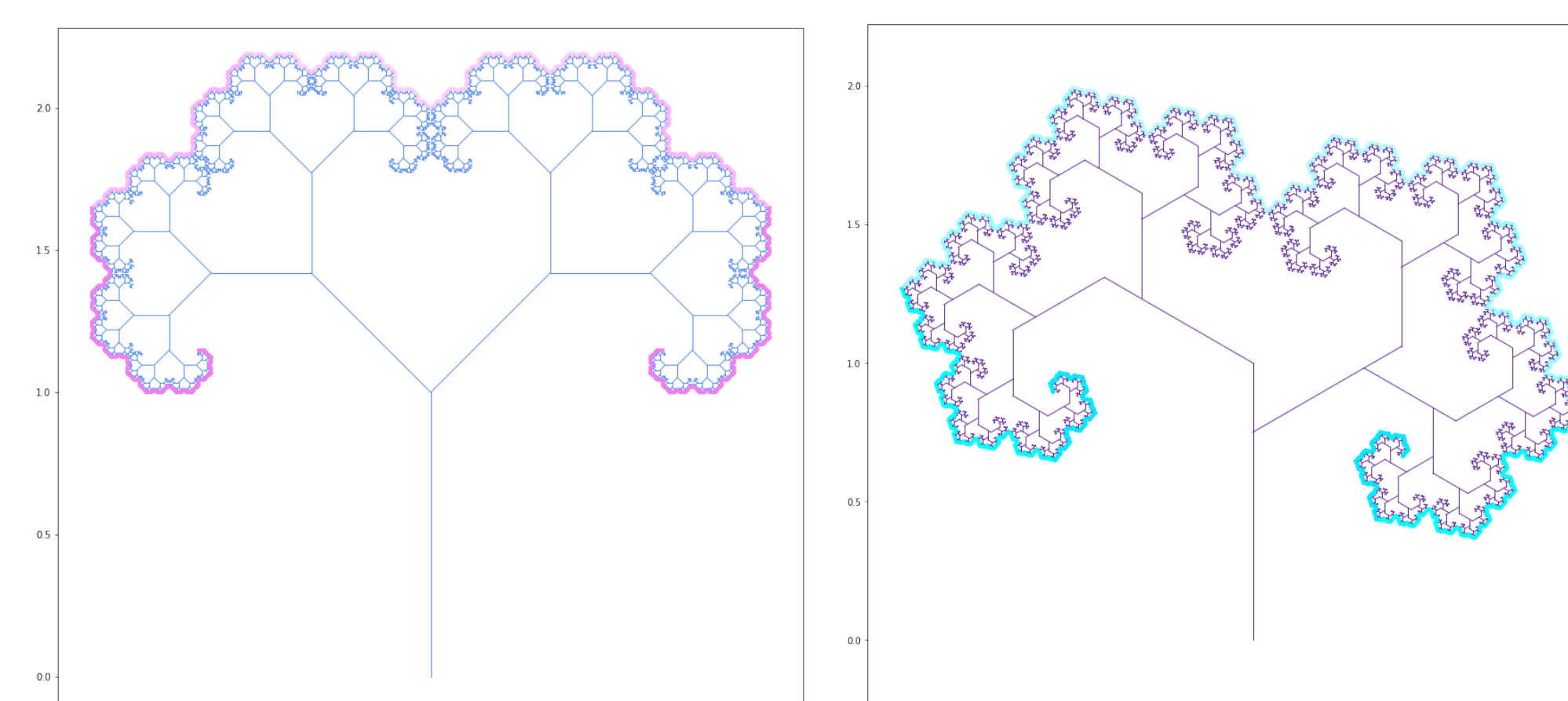

Figure 8:Koch Canopies: $\theta = \frac{\pi}{4}, s = 1; \theta = \frac{\pi}{3}, s = 0.75$

Dragon Curves
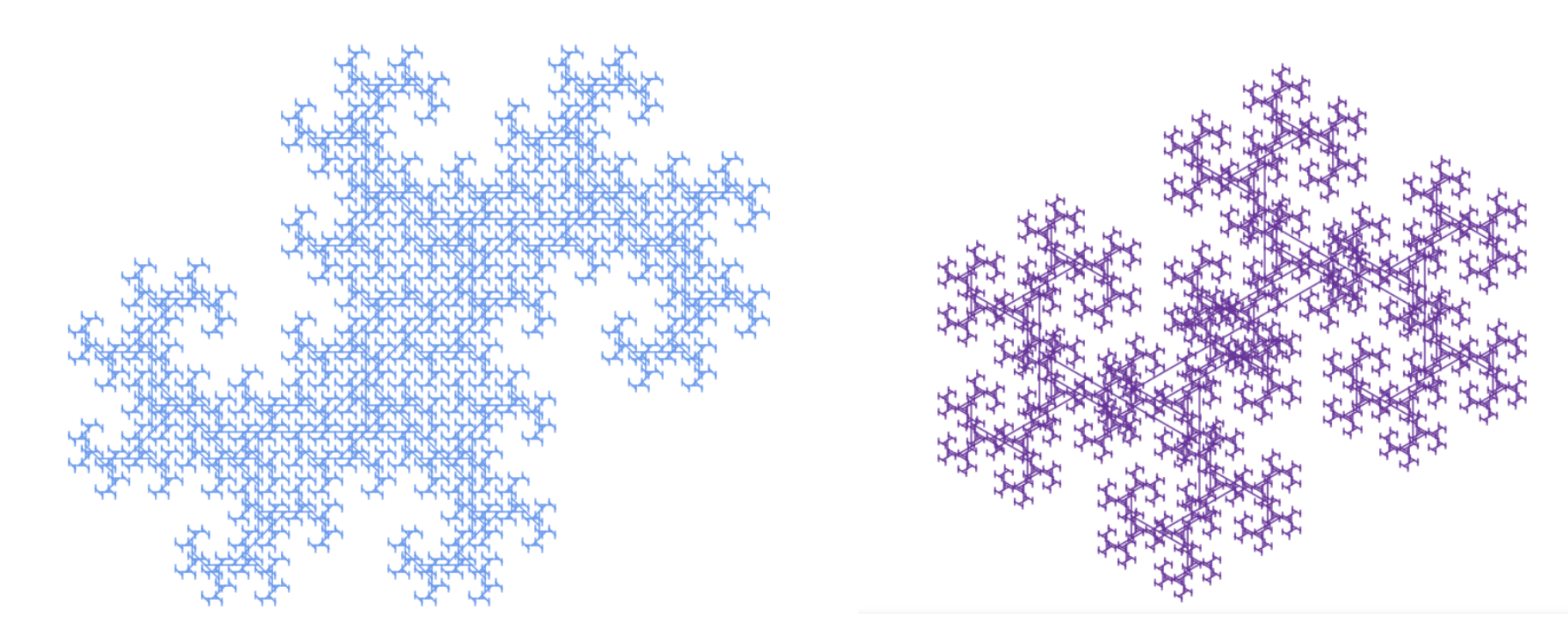

Figure 9:Dragon Curves: $\theta = \frac{\pi}{4}, \theta = \frac{\pi}{3}$

Other asymmetric trees we looked at were those with two different angles, one for the right turns and another for the left. This yielded fruitful results when the angles were supplementary. We found self-contact in both the $\frac{\pi}{4}, \frac{3\pi}{4}$ tree and the $\frac{\pi}{3}, \frac{2\pi}{3}$ tree (Figure 9). The $\frac{\pi}{4}$ tree's self contact point is at $(0,1)$, which means every point created by the iterated function system is in the fractal canopy. This is a good indicator that the tree is space filling, or contains all of the real points within its geometry.

Space Filling Proof

The first step is to show that the line created by the first left and the first right is complete, or that all x-coordinates on the line are present. We developed an algorithm based on a pattern of four turn sequences to approach any real value on the line. The rest of the fractal is made up of lines (Figure 10) created by line segments self-similar to the first filled line, meaning that every line segment is also complete. The distance between these segments shrinks by $\frac{1}{\sqrt{2}}$ with each iteration, causing the gaps to converge to zero and create complete, filled-in lines across the fractal. The distance between these lines shrinks by the same amount, causing the infinite lines to get as close as possible to each other and to fill the fractal completely.
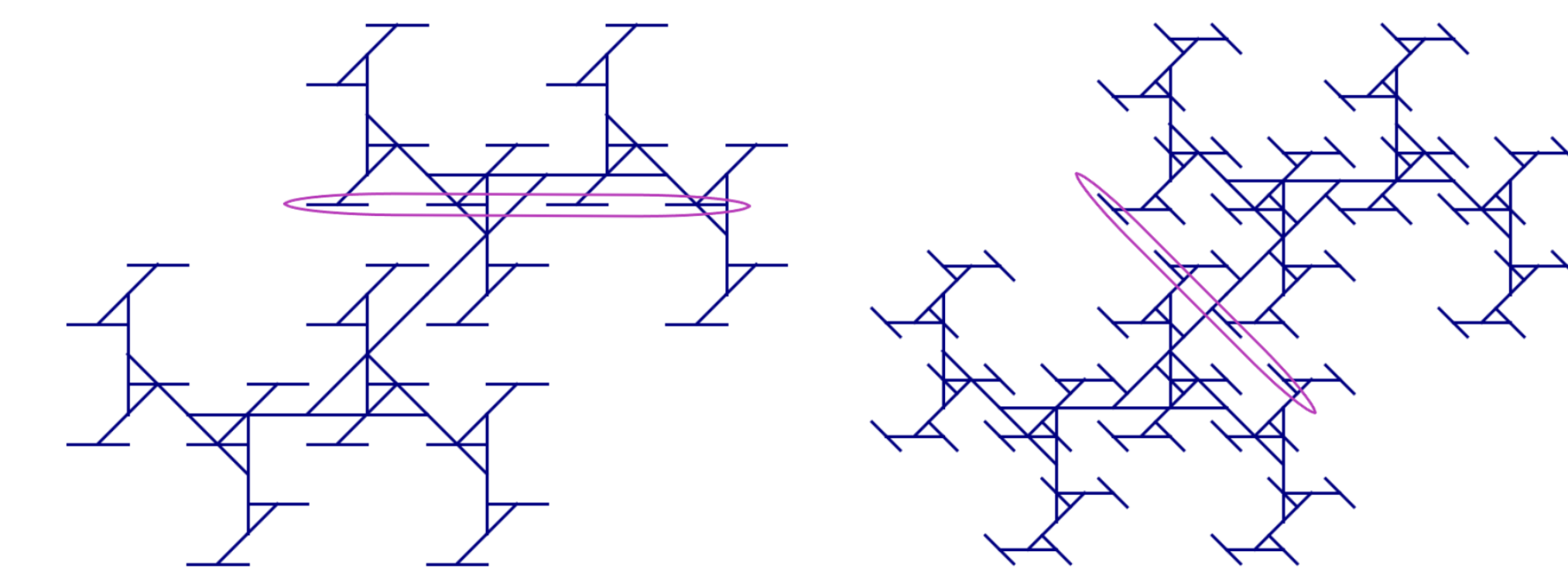

Figure 10:Lines: $\theta = \frac{\pi}{4}$

Acknowledgements