

**Parseval-Goldstein Type Identities Involving  
the  $\mathcal{F}_{S,2}$ -Transform, the  $\mathcal{F}_{C,2}$ -Transform  
and the  $\mathcal{P}_4$ -Transform and Their Applications**

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**Abstract**

In the present paper the authors consider several new integral transforms including the  $\mathcal{F}_{S,2}$ -transform, the  $\mathcal{F}_{C,2}$ -transform and the  $\mathcal{P}_4$ -transform as generalizations of the classical Fourier sine transform, the classical Fourier cosine transform, and the classical Stieltjes transform, respectively. Many identities involving these transforms are given. By making use of these identities, a number of new Parseval-Goldstein type identities are obtained for these and many other well-known integral transforms. The identities proven in this paper are shown to give rise to useful corollaries for evaluating infinite integrals of special functions. Some example are also given as illustrations of the results presented here.

**2000 Mathematics Subject Classification.** Primary 44A10, 44A15; Secondary 33C10, 44A35.

**Key Words and Phrases.** Laplace transforms, Widder potential transforms,  $\mathcal{L}_2$ -transforms,  $\mathcal{F}_{S,2}$ -transforms,  $\mathcal{F}_{C,2}$ -transform,  $\mathcal{P}_4$ -transforms, Stieltjes transforms,  $K$ -transforms, Hankel transforms, Fourier sine transforms, Fourier cosine transforms, Parseval-Goldstein type theorems.

**1. Introduction, Definitions and Preliminaries**

The following Laplace-type transform:

$$\mathcal{L}_2\{f(x); y\} = \int_0^{\infty} x \exp(-x^2 y^2) f(x) dx \quad (1)$$

was introduced by Sadek and Yürekli [18]. Subsequently, various Parseval-Goldstein type identities were given in (for example) [1], [2], [5], [15], and [16] for the  $\mathcal{L}_2$ -transform. New solutions techniques were obtained for the Bessel differential equation in [20] and the Hermite differential equation in [21] using this integral transform. The  $\mathcal{L}_2$ -transform is related to the classical Laplace transform:

$$\mathcal{L}\{f(x); y\} = \int_0^{\infty} \exp(-xy) f(x) dx, \quad (2)$$

by means of the following relationship:

$$\mathcal{L}_2\{f(x); y\} = \frac{1}{2} \mathcal{L}\{f(x^{1/2}); y^2\}. \quad (3)$$

There are numerous analogous results in the literature on integral transforms (see, for instance, [11], [14], [17] and [19]).

Over four decades ago, Widder [13] presented a systematic account of so-called the Widder potential-transform:

$$\mathcal{P}\{f(x); y\} = \int_0^{\infty} \frac{x f(x)}{x^2 + y^2} dx. \quad (4)$$

which, by an exponential change of variables, becomes a convolution transform with kernel belonging to a general class investigated by Hirschman and Widder [7]. Subsequently, Srivastava and Singh [9] gave the following Parseval-Goldstein type identity

$$\int_0^{\infty} y \mathcal{P}\{f(x); y\} g(y) dy = \int_0^{\infty} x \mathcal{P}\{g(y); x\} f(x) dx. \quad (5)$$

The Widder potential transform (4) is related to the Stieltjes transform as follows (see also [11] and [12]):

$$\mathcal{P}\{f(x); y\} = \frac{1}{2} \mathcal{S}\{f(x^{1/2}); y^2\}, \quad (6)$$

where the Stieltjes transform is defined by

$$\mathcal{S}\{f(x); y\} = \int_0^{\infty} \frac{f(x)}{x + y} dx. \quad (7)$$

In this article, we introduce several new integral transforms and establish potentially useful identities for these integral transforms. First of all, the  $\mathcal{P}_4$ -transform is defined by

$$\mathcal{P}_4\{f(x); y\} = \int_0^{\infty} \frac{x^3 f(x)}{x^4 + y^4} dx, \quad (8)$$

which is related to the Stieltjes transform (7) as follows:

$$\mathcal{P}_4\{f(x); y\} = \frac{1}{4} \mathcal{S}\{f(x^{1/4}); y^4\}, \quad (9)$$

and the Widder potential transform (4) as follows:

$$\mathcal{P}_4\{f(x); y\} = \frac{1}{2} \mathcal{P}\{f(x^{1/2}); y^2\}. \quad (10)$$

The  $\mathcal{F}_{S,2}$ -transform is defined by

$$\mathcal{F}_{S,2}\{f(x); y\} = \int_0^{\infty} x \sin(x^2 y^2) f(x) dx, \quad (11)$$

which is related to the Fourier-sine transform:

$$\mathcal{F}_S\{f(x); y\} = \int_0^\infty \sin(xy) f(x) dx, \quad (12)$$

by means of the following identity:

$$\mathcal{F}_{S,2}\{f(x); y\} = \frac{1}{2} \mathcal{F}_S\{f(x^{1/2}); y^2\} \quad (13)$$

Similarly, the  $\mathcal{F}_{C,2}$ -transform is defined by

$$\mathcal{F}_{C,2}\{f(x); y\} = \int_0^\infty x \cos(x^2 y^2) f(x) dx, \quad (14)$$

which is related to the Fourier-cosine transform:

$$\mathcal{F}_C\{f(x); y\} = \int_0^\infty \cos(xy) f(x) dx, \quad (15)$$

by means of the following identity:

$$\mathcal{F}_{C,2}\{f(x); y\} = \frac{1}{2} \mathcal{F}_C\{f(x^{1/2}); y^2\} \quad (16)$$

The  $\mathcal{H}_{\nu,2}$ -transform is defined by

$$\mathcal{H}_{\nu,2}\{f(x); y\} = \int_0^\infty (x^2 y) J_\nu(x^2 y^2) f(x) dx, \quad (17)$$

which is related to the  $\mathcal{H}_\nu$ -transform:

$$\mathcal{H}_\nu\{f(x); y\} = \int_0^\infty (xy)^{1/2} J_\nu(x^2 y^2) f(x) dx, \quad (18)$$

by means of the following identity:

$$\mathcal{H}_{\nu,2}\{f(x); y\} = \frac{1}{2} \mathcal{H}_\nu\{f(x^{1/2}); y^2\}. \quad (19)$$

The  $\mathcal{K}_{\nu,2}$ -transform is defined by

$$\mathcal{K}_{\nu,2}\{f(x); y\} = \int_0^\infty (x^2 y) K_\nu(x^2 y^2) f(x) dx, \quad (20)$$

which is related to the  $\mathcal{K}_\nu$ -transform:

$$\mathcal{K}_\nu\{f(x); y\} = \int_0^\infty (xy)^{1/2} K_\nu(x^2 y^2) f(x) dx, \quad (21)$$

by means of the following identity:

$$\mathcal{K}_{\nu,2}\{f(x); y\} = \frac{1}{2} \mathcal{K}_\nu\{f(x^{1/2}); y^2\}. \quad (22)$$

In Section 2 of this paper, we derive several identities involving the  $\mathcal{F}_{S,2}$ -transform, the  $\mathcal{F}_{C,2}$ -transform, and the  $\mathcal{P}_4$ -transform. Using these identities, a number of new Parseval-Goldstein type identities are then obtained for these and many other well-known integral transforms. Our main theorem is shown to yield new identities for the integral transforms introduced above. As applications of the identities (contained in the Lemma) and the Theorem, some illustrative examples are also given.

## 2. The Main Parseval-Goldstein Type Theorem

In the following lemma, we give identities involving the classical Laplace transform (2),  $\mathcal{F}_{S,2}$ -transform (11),  $\mathcal{F}_{C,2}$ -transform (14), and  $\mathcal{P}_4$ -transform (8).

**Lemma.** *The identities*

$$\mathcal{L}_2\left\{\mathcal{F}_{S,2}\{f(x); u\}; y\right\} = \frac{1}{2} \mathcal{P}_4\{f(x); y\}, \quad (23)$$

$$\mathcal{F}_{S,2}\left\{\mathcal{L}_2\{f(x); u\}; y\right\} = \frac{y^2}{2} \mathcal{P}_4\left\{\frac{f(x)}{x^2}; y\right\}, \quad (24)$$

$$\mathcal{L}_2\left\{\mathcal{F}_{C,2}\{f(x); u\}; y\right\} = \frac{y^2}{2} \mathcal{P}_4\left\{\frac{f(x)}{x^2}; y\right\}, \quad (25)$$

$$\mathcal{F}_{C,2}\left\{\mathcal{L}_2\{f(x); u\}; y\right\} = \frac{1}{2} \mathcal{P}_4\{f(x); y\}, \quad (26)$$

$$\mathcal{L}_2\left\{\mathcal{F}_{S,2}\{f(x); u\}; y\right\} = \mathcal{F}_{C,2}\left\{\mathcal{L}_2\{f(x); u\}; y\right\}, \quad (27)$$

and

$$\mathcal{F}_{S,2}\left\{\mathcal{L}_2\{f(x); u\}; y\right\} = \mathcal{L}_2\left\{\mathcal{F}_{C,2}\{f(x); u\}; y\right\} \quad (28)$$

hold true, provided that each member of the assertions (23)–(26) exists.

*Proof.* We only give the proof of (23). The proofs of (24), (25), and (26) are similar to the proof of (23). The proof of (27) follows from (23) and (26) and the proof of (28) follows from (24) and (25).

In order to prove (23), we begin by using the definitions of the  $\mathcal{L}_2$ -transform (1) and the  $\mathcal{F}_{S,2}$ -transform (11). We thus observe that

$$\begin{aligned} \mathcal{L}_2\left\{\mathcal{F}_{S,2}\{f(x); u\}; y\right\} &= \int_0^\infty u \exp(-u^2 y^2) \\ &\quad \times \left[ \int_0^\infty x \sin(u^2 x^2) f(x) dx \right] du. \end{aligned} \quad (29)$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, it follows from (29) that

$$\begin{aligned} &\mathcal{L}_2\left\{\mathcal{F}_{S,2}\{f(x); u\}; y\right\} \\ &= \int_0^\infty x f(x) \left[ \int_0^\infty u \exp(-u^2 y^2) \sin(u^2 x^2) du \right] dx \end{aligned} \quad (30)$$

Next, by evaluating the inner integral on the right-hand side of (30), we obtain

$$\mathcal{L}_2\left\{\mathcal{F}_{S,2}\{f(x); u\}; y\right\} = \frac{1}{2} \int_0^\infty \frac{x^3 f(x)}{x^4 + y^4} dx. \quad (31)$$

Finally, by using the definition (8) of the  $\mathcal{P}_4$ -transform on the right hand-side of (31), we deduce the identity (23) asserted by the Lemma.

**Example 1.** We show that

$$\mathcal{P}_4\{x^\nu J_\nu(ax); y\} = \frac{1}{2} y^\nu [\omega^\nu K_\nu(a\omega y) + \bar{\omega}^\nu K_\nu(a\bar{\omega} y)] \quad (32)$$

and

$$\mathcal{P}_4\{x^{\nu-2} J_\nu(ax); y\} = \frac{i}{2} y^{\nu-2} [\omega^\nu K_\nu(a\omega y) - \bar{\omega}^\nu K_\nu(a\bar{\omega} y)] \quad (33)$$

where

$$\Re(\nu) > -1, \quad |\arg a| < \pi/4, \quad \text{and} \quad \omega = \exp\left(\frac{i\pi}{4}\right). \quad (34)$$

*Demonstration.* If We first set

$$f(x) = x^\nu J_\nu(ax) \quad (35)$$

in the assertion (26) of the Lemma, and then use the identity (3) as well as a known formula [3, p. 185, Entry (30)], we find that

$$\begin{aligned} \mathcal{L}_2\{x^\nu J_\nu(ax); u\} &= \frac{1}{2} \mathcal{L}\{x^{\nu/2} J_\nu(ax^{1/2}); u^2\} \\ &= \frac{1}{2} \left(\frac{a}{2}\right)^\nu u^{-2\nu-2} \exp(-a^2/4u^2). \end{aligned} \quad (36)$$

Now we apply  $\mathcal{F}_{C,2}$ -transform to (13) and make use the relation (1.16). We thus obtain

$$\begin{aligned} \mathcal{F}_{C,2}\left\{\mathcal{L}_2\{x^\nu J_\nu(ax); u\}; y\right\} \\ &= \frac{1}{2} \left(\frac{a}{2}\right)^\nu \mathcal{F}_{C,2}\left\{u^{-2\nu-2} \exp(-a^2/4u^2); y\right\} \\ &= \frac{1}{4} \left(\frac{a}{2}\right)^\nu \mathcal{F}_C\left\{u^{-\nu-1} \exp(-a^2/4u); y^2\right\} \end{aligned} \quad (37)$$

Making use of another known formula [3, p. 16, Entry (21), ], the Fourier cosine transform on the right-hand side of (36) is given by

$$\begin{aligned} \mathcal{F}_C\left\{u^{-\nu-1} \exp(-a^2/4u); y^2\right\} \\ &= \left(\frac{2}{a}\right)^\nu y^\nu [\omega^\nu K_\nu(a\omega y) + \bar{\omega}^\nu K_\nu(a\bar{\omega} y)] \\ &\quad \left(\omega = \exp(i\pi/4)\right). \end{aligned} \quad (38)$$

The assertion (32) of Example 1 follows upon substituting the result (37) into (36) and then using the identity (26) of the Lemma.

Similarly, in order to prove (33), we substitute (34) into the identity (24) of the Lemma. By applying the  $\mathcal{F}_{S,2}$ -transform to (35) and using the relation (13), we find that

$$\begin{aligned} \mathcal{F}_{S,2}\left\{\mathcal{L}_2\{x^\nu J_\nu(ax); u\}; y\right\} \\ &= \frac{1}{4} \left(\frac{a}{2}\right)^\nu \mathcal{F}_S\left\{u^{-\nu-1} \exp(-a^2/4u); y^2\right\} \end{aligned} \quad (39)$$

Now, by using the known formula [3, p. 75, Entry (31)] to evaluate the Fourier sine transform on the right-hand side of (38), as well as the assertion (24) of the Lemma, we arrive at (33).

**Example 2.** We show that

$$\begin{aligned} \mathcal{L}_2 \left\{ u^{-2\nu-2} \sin \left( \frac{a^2}{4u^2} - \frac{\nu\pi}{2} \right); y \right\} \\ = \frac{i}{2} \left( \frac{2}{a} \right)^\nu y^\nu [\omega^\nu K_\nu(a\omega y) - \bar{\omega}^\nu K_\nu(a\bar{\omega} y)] \end{aligned} \quad (40)$$

and

$$\begin{aligned} \mathcal{L}_2 \left\{ u^{-2\nu-2} \cos \left( \frac{a^2}{4u^2} - \frac{\nu\pi}{2} \right); y \right\} \\ = \frac{1}{2} \left( \frac{2}{a} \right)^\nu y^\nu [\omega^\nu K_\nu(a\omega y) + \bar{\omega}^\nu K_\nu(a\bar{\omega} y)] \end{aligned} \quad (41)$$

where

$$-1 < \Re(\nu) < \frac{1}{2}, \quad |\arg a| < \frac{\pi}{4}, \quad \text{and} \quad \omega = \exp \left( \frac{i\pi}{4} \right).$$

*Demonstration.* If we set

$$f(x) = x^\nu J_\nu(ax) \quad (42)$$

in the Lemma and then use the identity (16) and the known formula [3, p. 54, Entry (26)], we have

$$\mathcal{F}_{C,2} \left\{ x^\nu J_\nu(ax); u \right\} = \frac{1}{2} \mathcal{F}_C \left\{ x^{\nu/2} J_\nu(ax^{1/2}); u^2 \right\}$$

Now we apply  $\mathcal{L}_2$ -transform to (42) and use the relation (25) of the Lemma together with the formula (33) of the Example 1, we arrive at the assertion (39) of Example 2.

Similarly, in order to demonstrate the assertion (40) of Example 2, we substitute  $f(x)$  from (41) into the relationship (23) of the Lemma. We then use the identity (13) and the known formula [3, p. 110, Entry (28)] to obtain the assertion (40) of Example 2.

We now state and prove our main results contained in the following Theorem. **Theorem.**

The following Parseval-Goldstein type identities hold true:

$$\begin{aligned} \int_0^\infty y \mathcal{L}_2 \{f(x); y\} \mathcal{F}_{S,2} \{g(u); y\} dy \\ = \frac{1}{2} \int_0^\infty x f(x) \mathcal{P}_4 \{g(u); x\} dx, \end{aligned} \quad (43)$$

$$\begin{aligned} \int_0^\infty y \mathcal{L}_2 \{f(x); y\} \mathcal{F}_{S,2} \{g(u); y\} dy \\ = \frac{1}{2} \int_0^\infty u^3 g(u) \mathcal{P}_4 \left\{ \frac{f(x)}{x^2}; u \right\} du, \end{aligned} \quad (44)$$

$$\begin{aligned} \int_0^\infty y \mathcal{L}_2 \{f(x); y\} \mathcal{F}_{C,2} \{g(u); y\} dy \\ = \frac{1}{2} \int_0^\infty u g(u) \mathcal{P}_4 \{f(x); u\} du, \end{aligned} \quad (45)$$

$$\begin{aligned} \int_0^\infty y \mathcal{L}_2 \{f(x); y\} \mathcal{F}_{C,2} \{g(u); y\} dy \\ = \frac{1}{2} \int_0^\infty x^3 f(x) \mathcal{P}_4 \left\{ \frac{g(u)}{u^2}; x \right\} dx, \end{aligned} \quad (46)$$

and

$$\begin{aligned} \int_0^\infty x f(x) \mathcal{P}_4 \{g(u); x\} dx \\ = \int_0^\infty u^3 g(u) \mathcal{P}_4 \left\{ \frac{f(x)}{x^2}; u \right\} du \end{aligned} \quad (47)$$

provided that the integrals involved converge absolutely.

*Proof.* We only give the proof of the Parseval-Goldstein identity (43) because the proof of (44), (45), and (46) are similar and the identity (47) follows easily from the assertions (43) and (44), or from the assertions (45) and (46). Indeed, by the definition (13) of the  $\mathcal{F}_{S,2}$ -transform, we have

$$\begin{aligned} \int_0^\infty y \mathcal{L}_2 \{f(x); y\} \mathcal{F}_{S,2} \{g(u); y\} dy \\ = \int_0^\infty y \mathcal{F}_{S,2} \{g(u); y\} \left[ \int_0^\infty x \exp(-x^2 y^2) f(x) dx \right] dy. \end{aligned} \quad (48)$$

Changing the order of integration (which is permissible by absolute convergence of the integrals involved) and using the definition (1) of  $\mathcal{L}_2$ -transform once more, we find from

(48) that

$$\begin{aligned}
& \int_0^\infty y \mathcal{L}_2 \{f(x); y\} \mathcal{F}_{S,2} \{g(u); y\} dy \\
&= \int_0^\infty x f(x) \left[ \int_0^\infty y \exp(-x^2 y^2) \mathcal{F}_{S,2} \{g(u); y\} dy \right] dx \\
&= \int_0^\infty x f(x) \mathcal{L}_2 \left\{ \mathcal{F}_{S,2} \{g(u); y\}; x \right\} dx.
\end{aligned} \tag{49}$$

Now the assertion (43) follows from (49) and (23) of the Lemma.

**Corollary 1** *The following identities hold true for the  $\mathcal{L}_2$ -transform (1), the  $\mathcal{P}_4$ -transform (8), and the  $\mathcal{F}_{S,2}$ -transform (11):*

$$\mathcal{P}_4 \left\{ \mathcal{F}_{S,2} \{g(u); y\}; t \right\} = \frac{\pi}{4} \mathcal{L}_2 \{g(u); t\}, \tag{50}$$

and

$$\mathcal{F}_{C,2} \left\{ \mathcal{P}_4 \{g(u); x\}; y \right\} = \frac{\pi}{4} \mathcal{L}_2 \{g(u); y\} \tag{51}$$

provided that each member of the assertions (50) and (51) exists.

*Proof.* We set

$$f(x) = \cos(t^2 x^2) \tag{52}$$

in the above Theorem. By making the identity (3) and the known formula [3, p. 154, Entry (43)], we obtain

$$\mathcal{L}_2 \left\{ \cos(t^2 x^2); y \right\} = \frac{1}{2} \mathcal{L} \left\{ \cos(t^2 x); y^2 \right\} = \frac{1}{2} \frac{y^2}{y^4 + t^4}. \tag{53}$$

Now, applying the known formula [2, p. 401, Eq. (32)], we have

$$\mathcal{P}_4 \left\{ x^{-2} \cos(t^2 x^2); u \right\} = \frac{\pi}{4} u^{-2} \exp(-t^2 u^2). \tag{54}$$

The assertion (50) follows upon substituting the results (53) and (54) into the Parseval-Goldstein type identity (44). The assertion (51) is obtained directly using the formula (54).

**Remark 1** The Parseval-Goldstein type identity (51) is obtained earlier by *et al.* [2, p. 404, (67)].

**Corollary 2** *The following inversion identities hold true for the  $\mathcal{F}_{S,2}$ -transform (11) and the  $\mathcal{F}_{C,2}$ -transform (14):*

$$\mathcal{F}_{S,2} \left\{ \mathcal{F}_{S,2} \{g(u); y\}; t \right\} = \frac{\pi}{8} g(t), \tag{55}$$

and

$$\mathcal{F}_{C,2} \left\{ \mathcal{F}_{C,2} \{g(u); y\}; t \right\} = \frac{\pi}{8} g(t), \tag{56}$$

provided that each member of the assertions (55) and (56) exists.

*Proof.* If we put

$$f(x) = \mathcal{F}_{S,2}\{g(u); x\}. \quad (57)$$

in the identity (23) asserted by the Lemma, we have

$$\mathcal{L}_2\left\{\mathcal{F}_{S,2}\left\{\mathcal{F}_{S,2}\{g(u); x\}; y\right\}; t\right\} = \frac{1}{2} \mathcal{P}_4\left\{\mathcal{F}_{S,2}\{g(u); x\}; t\right\}. \quad (58)$$

Now, using the identity (50) of the Corollary 1, we arrive at the result (55). Similarly, the proof of the assertion (56) follows from the identities (26) of the Lemma and the identity (51) of the Corollary 1.

**Corollary 3** *The following identities hold true for the  $\mathcal{F}_{S,2}$ -transform (11), the  $\mathcal{F}_{C,2}$ -transform (14), the  $\mathcal{L}_2$ -transform (1), and the  $\mathcal{P}_4$ -transform (8):*

$$\mathcal{P}_4\left\{\mathcal{F}_{S,2}\{g(u); y\}; t\right\} = \mathcal{F}_{C,2}\left\{\mathcal{P}_4\{g(u); x\}; t\right\} \quad (59)$$

$$\mathcal{P}_4\left\{\frac{1}{y^2} \mathcal{F}_{S,2}\{g(u); y\}; t\right\} = \frac{1}{t^2} \mathcal{F}_{S,2}\left\{\mathcal{P}_4\{g(u); x\}; t\right\} \quad (60)$$

and

$$\mathcal{P}_4\left\{u^2 \mathcal{P}_4\left\{\frac{f(x)}{x^2}; u\right\}; t\right\} = \frac{\pi}{2} \mathcal{L}_2\left\{\mathcal{L}_2\{f(x); y\}; t\right\} \quad (61)$$

provided that each member of the assertions (59), (60) and (61) exists.

*Proof.* The identity (59) follows easily from the assertions (50) and (51). To verify the assertion (60), we set

$$f(x) = \sin(t^2 x^2) \quad (62)$$

in the assertion (43) of our Theorem. Using the identity (3) and making use of the known formula [3, p. 150, Entry (1)], we obtain

$$\mathcal{L}_2\left\{\sin(t^2 x^2); y\right\} = \frac{1}{2} \mathcal{L}\left\{\sin(t^2 x); y^2\right\} = \frac{1}{2} \frac{t^2}{t^4 + y^4}. \quad (63)$$

Similarly, the identity (61) is proved by setting

$$g(u) = \frac{u^2}{u^4 + t^4} \quad (64)$$

in the assertion (44) of our Theorem. and using the identity (13) and the known formula [3, p. 65, Entry (15)].

**Corollary 4** *The following identity hold true for the  $\mathcal{H}_{\nu,2}$ -transform (17) and the  $\mathcal{K}_{\nu,2}$ -transform (20):*

$$\mathcal{H}_{\nu,2}\left\{u^{2\nu+1} \mathcal{P}_4\left\{\frac{f(x)}{x^2}; u\right\}; t\right\} = \frac{1}{2} \mathcal{K}_{\nu,2}\left\{x^{2\nu-1} f(x); t\right\} \quad (65)$$

provided that each member of the assertions (65).

*Proof.* If we put

$$g(u) = u^{2\nu} J_\nu(t^2 u^2) \quad (66)$$

in the identity (47) of the Theorem and use the following known formula [2, p. 402, Equation (39)]

$$\mathcal{P}_4\{u^{2\nu} J_\nu(t^2 u^2); x\} = \frac{1}{2} x^{2\nu} K_\nu(t^2 x^2). \quad (67)$$

we arrive at the assertion (65) of Corollary 4 upon substituting (66) and (67) into the assertion (47) our Theorem.

**Remark 2.** By using the techniques applied in Corollaries 1 to 4 above it is possible to obtain new identities similar to the identities presented in the Corollaries themselves for  $\mathcal{F}_{S,2}$ -transform (11), the  $\mathcal{F}_{C,2}$ -transform (14), the  $\mathcal{L}_2$ -transform (1), and the  $\mathcal{P}_4$ -transform (8) (see Section 3 below).

### 3. Another Set of Useful Corollaries

Several interesting Parseval-Goldstein type corollaries involving  $\mathcal{F}_{S,2}$ -transform (11), the  $\mathcal{F}_{C,2}$ -transform (14), the  $\mathcal{L}_2$ -transform (1), and the  $\mathcal{P}_4$ -transform (8) will be presented in this section as *further* applications of our main Theorem.

**Corollary 5.** *The following Parseval-Goldstein type identities hold true:*

$$\int_0^\infty y^{-2\nu+1} \mathcal{F}_{S,2}\{g(u); y\} dy = \frac{1}{\Gamma(\nu)} \int_0^\infty x^{2\nu-1} \mathcal{P}_4\{g(u); x\} dx, \quad (68)$$

$$\int_0^\infty y^{-2\nu+1} \mathcal{F}_{C,2}\{g(u); y\} dy = \frac{1}{\Gamma(\nu)} \int_0^\infty x^{2\nu+1} \mathcal{P}_4\left\{\frac{g(u)}{u^2}; x\right\} dx, \quad (69)$$

$$\int_0^\infty y^{-2\nu+1} \mathcal{F}_{S,2}\{g(u); y\} dy = \frac{\pi}{4\Gamma(\nu)} \csc\left(\frac{\nu\pi}{2}\right) \int_0^\infty u^{2\nu-1} g(u) du, \quad (70)$$

$$\int_0^\infty y^{-2\nu+1} \mathcal{F}_{C,2}\{g(u); y\} dy = \frac{\pi}{4\Gamma(\nu)} \sec\left(\frac{\nu\pi}{2}\right) \int_0^\infty u^{2\nu-1} g(u) du, \quad (71)$$

*provided that the integrals involved converge absolutely.*

*Proof.* We put

$$f(x) = x^{2\nu-2} \quad (72)$$

in the assertion (43) of our Theorem. If we use the identity (3), we shall find that

$$\mathcal{L}_2\{x^{2\nu-2}; y\} = \frac{1}{2} \mathcal{L}\{x^{\nu-1}; y^2\} = \frac{\Gamma(\nu)}{2y^{2\nu}} \quad (\Re(\nu) > 0; y \in \mathbb{R}^+). \quad (73)$$

Now, by applying the identity (10), and use the known formula [14, p. 248, (A1)], we find that

$$\mathcal{P}_4\{x^{2\nu-2}; y\} = \frac{1}{2} \mathcal{P}\{x^{\nu-1}; y^2\} = \frac{\pi}{4} \sec\left(\frac{\nu\pi}{2}\right) y^{2\nu-2}. \quad (74)$$

The assertions (68) and (69) are obtained upon inserting (73) and (74) into (43) and (46), respectively. Similarly, the assertions (70) and (71) follows easily upon substituting for  $f(x)$  from (72) into the identities (44) and (45) of our Theorem.

**Corollary 6** *The following Parseval-Goldstein type identities involving  $\mathcal{P}_4$ -transform (8) hold true:*

$$\int_0^\infty u^{2\nu+1} g(u) du = \frac{4}{\pi} \cos\left(\frac{\nu\pi}{2}\right) \int_0^\infty x^{2\nu+1} \mathcal{P}_4\{g(u); x\} dx, \quad (75)$$

$$\int_0^\infty x^{2\nu-1} f(x) dx = \frac{4}{\pi} \cos\left(\frac{\nu\pi}{2}\right) \int_0^\infty u^{2\nu+1} \mathcal{P}_4\left\{\frac{f(x)}{x^2}; u\right\} du, \quad (76)$$

*provided that the integrals involved converge absolutely.*

*Proof.* The assertion (75) easily follows by setting  $f(x) = x^{2\nu}$  in the assertion (47) of the main Theorem and using the known formula [14, p. 248, (A1)]. The proof of the second assertion (76) of Corollary 6 is similar.

**Corollary 7** *The following Parseval-Goldstein type identities involving  $\mathcal{L}_2$ -transform (1) and  $\mathcal{P}_4$ -transform (8) hold true:*

$$\begin{aligned} \int_0^\infty x^{2\nu-1} f(x) dx \\ = \frac{2}{\Gamma(1-\nu)} \int_0^\infty y^{-2\nu+1} \mathcal{L}_2\{f(x); y\} dy \end{aligned} \quad (77)$$

$$\begin{aligned} \int_0^\infty y^{-2\nu+1} \mathcal{L}_2\{f(x); y\} dy \\ = \frac{2}{\pi} \Gamma(1-\nu) \sin\left(\frac{\nu\pi}{2}\right) \int_0^\infty u^{2\nu-1} \mathcal{P}_4\{f(x); u\} dy, \end{aligned} \quad (78)$$

$$\begin{aligned} \int_0^\infty y^{-2\nu+1} \mathcal{L}_2\{f(x); y\} dy \\ = \frac{2}{\pi} \Gamma(1-\nu) \cos\left(\frac{\nu\pi}{2}\right) \int_0^\infty u^{2\nu+1} \mathcal{P}_4\left\{\frac{f(x)}{x^2}; u\right\} du, \end{aligned} \quad (79)$$

*provided that the integrals involved converge absolutely and  $0 < \Re(\nu) < 1$ .*

*Proof.* We first put

$$g(u) = u^{2\nu-2} \quad (80)$$

in the assertion (43) of our Theorem. Then, if we use the identity (13) and the known formula [6, p. 430, Entry 3.761-4], we find that

$$\mathcal{F}_{S,2}\{u^{2\nu-2}; y\} = \frac{1}{2} \mathcal{F}_S\{u^{\nu-1}; y^2\} = \frac{\pi}{4\Gamma(1-\nu)} \sec\left(\frac{\nu\pi}{2}\right) y^{-2\nu}. \quad (81)$$

Using the results (74) and (81) in (43), we arrive at the assertion (77) of Corollary 7. The proofs of the assertions (78) and (79) of Corollary 7 are obtained easily upon using (respectively) the identity (16) and the known formula [6, p. 430, Entry 3.761-9] in the identities (44) and (45) of our Theorem.

#### 4. Further Illustrative Examples

An interesting illustration for the identity (50) asserted by Corollary 1 is contained in the following example.

**Example 3.** We show that

$$\mathcal{F}_{C,2}\{x^{2\nu} K_\nu(t^2 x^2); y\} = 2^{\nu-2} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) t^{2\nu} (y^4 + t^4)^{-\nu-1/2} \quad (82)$$

$$\left(\Re(\nu) > -\frac{1}{2}\right).$$

*Demonstration.* If we set

$$g(u) = u^{2\nu} J_\nu(t^2 u^2). \quad (83)$$

in the assertion (51) of Corollary 1 and use the known formulas [2, p. 402, Equation (39)] and [3, p. 182, Entry (7)], we get

$$\mathcal{P}_4\{g(u); x\} = \frac{1}{2} x^{2\nu} K_\nu(t^2 x^2) \quad (84)$$

and

$$\mathcal{L}_2\{g(u); y\} = \frac{2^{\nu-1}}{\sqrt{\pi}} \Gamma\left(\nu + \frac{1}{2}\right) t^{2\nu} (y^4 + t^4)^{-\nu-1/2}, \quad (85)$$

respectively. The assertion (82) follows immediately upon substituting (84) and (85) into (51) of Corollary 1.

**Remark 3.** Using the identity (13) in the formula (82) of Example 3, we obtain the following consequence (cf. [3, p. 49, Entry (1)]):

$$\mathcal{F}_C\{x^\nu K_\nu(tx); y\} = 2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) t^\nu (y^2 + t^2)^{-\nu-1/2} \quad (86)$$

Another illustration of Corollary 1 is contained in the following example.

**Example 4:** We show that

$$\mathcal{F}_{C,2}\{\exp(-ax) \cos(ax); y\} = a (2\pi)^{1/2} (2y)^{-3} \exp\left(-\frac{a^2}{2y^2}\right). \quad (87)$$

*Demonstration.* If we put

$$g(u) = \sin(\sqrt{2} a u) \quad (88)$$

in the assertion (51) of Corollary 1 and use the known formulas [6, p. 420, Entry 3.727-10] and [3, p. 153, Entry (32)], we get

$$\mathcal{P}_4\{g(u); x\} = \frac{\pi}{2} \exp(-ax) \cos(ax) \quad (89)$$

and

$$\mathcal{L}_2\{g(u); y\} = \frac{a}{2y^3} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{a^2}{2y^2}\right), \quad (90)$$

respectively. Substituting (89) and (90) into (51) of Corollary 1, we arrive at the assertion (87).

**Remark 4** Using the identity (13) in the formula (88) of Example 4, we obtain the following consequence (cf. [8, p. 25, Entry (5.65)]):

$$\mathcal{F}_C\{\exp(-a\sqrt{x})\cos(a\sqrt{x});y\}=a\sqrt{\frac{\pi}{(2y)^3}}\exp\left(-\frac{a^2}{2y}\right). \quad (91)$$

We conclude this investigation by remarking that many other infinite integrals can be evaluated in this manner by applying the above Lemma, the above Theorem, and their various corollaries and consequences consider here.

### Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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