New identities for the Glasser transform and their applications

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Abstract

In the present paper the authors show that an iteration of the $\mathscr{L}_2$-transform by itself is a constant multiple of the Glasser transform. Using this iteration identity, a Parseval-Goldstein type theorem for $\mathscr{L}_2$-transform and the Glasser transform is given. By making use of these results a number of new Parseval-Goldstein type identities are obtained for these and many other well-known integral transforms. The identities proven in this paper are shown to give rise to useful corollaries for evaluating infinite integrals of special functions. Some examples are also given as illustration of the results presented here.

Key words: Laplace transforms, $\mathscr{L}_2$-transforms, Glasser transforms, Fourier sine transforms, Fourier cosine transforms, Hankel transforms, $\mathscr{K}$-transforms, $\mathscr{E}_1$-transforms, $\mathscr{E}_{2,1}$-transforms, Parseval-Goldstein type theorems.

2000 Mathematics Subject Classification. Primary 44A10, 44A15 ; Secondary 33C10, 44A35.

1 Introduction

Over a decade ago, Yürekli & Sadek (1991) presented a systematic account of so-called the $\mathscr{L}_2$-transform:

$$\mathscr{L}_2\Big\{f(x)\,;\,y\Big\} = \int_0^\infty x\,\exp\Big(-x^2\,y^2\Big)\,f(x)\,dx \tag{1.1}$$

The $\mathscr{L}_2$-transform is related to the classical Laplace transform

$$\mathscr{L}\Big\{f(x)\,;\,y\Big\} = \int_0^\infty \exp(-x\,y)\,f(x)\,dx \tag{1.2}$$

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by means of the following relationships:

$$\mathscr{L}_2\left\{f(x)\,;\,y\right\} = \frac{1}{2}\,\mathscr{L}\Big\{f\Big(\sqrt{x}\Big)\,;\,y^2\Big\}, \tag{1.3}$$

$$\mathscr{L}\left\{f(x)\,;\,y\right\} = 2\,\mathscr{L}_2\Big\{f\Big(x^2\Big)\,;\,\sqrt{y}\Big\}. \tag{1.4}$$

Subsequently, various Parseval-Goldstein type identities were given in (for example) Brown et al. (2007a), Brown et al. (2007b), Dernek et al. (2007), Glasser (1973), Yürekli (1999), and Yürekli (1999) for the $\mathscr{L}_2$-transform. New solutions techniques were obtained for the Bessel differential equation in Yürekli & Wilson (2002) and the Hermite differential equation in Yürekli & Wilson (2003) using this integral transform. There are numerous analogous results in the literature on various integral transforms (see, for instance Yürekli (1989), Srivastava & Yürekli (1995),Yürekli & Graziadio (1997), and Yürekli & Saygınsoy (1998)). Some of the results from Yürekli (1989) and Yürekli (1992) are applied to generalized functions by Adawi & Alawneh (2001).

Over three decades ago, Glasser (1973) considered so-called the Glasser transform

$$\mathscr{G}\Big\{f(x)\,;\,y\Big\} = \int_0^\infty \frac{f(x)}{\sqrt{x^2+y^2}}\,dx. \tag{1.5}$$

Glasser gave the following Parseval-Goldstein type theorem (cf. Glasser (1973, p. 171, Eq. (4)))

$$\int_0^\infty f(x)\,\mathscr{G}\Big\{g(y)\,;\,x\Big\}\,dx = \int_0^\infty g(x)\,\mathscr{G}\Big\{f(y)\,;\,x\Big\}\,dx, \tag{1.6}$$

and evaluated a number of infinite integrals involving Bessel functions. Additional results about the Glasser transform can be found in Srivastava & Yürekli (1995) and Kahramaner et al. (1995).

The Fourier sine transform and the Fourier cosine transform are defined as

$$\mathscr{F}_S\Big\{f(x)\,;\,y\Big\} = \int_0^\infty \sin(x\,y)\,f(x)\,dx, \tag{1.7}$$

and

$$\mathscr{F}_C\Big\{f(x)\,;\,y\Big\} = \int_0^\infty \cos(x\,y)\,f(x)\,dx, \tag{1.8}$$

respectively.

The Hankel transform is defined by

$$\mathscr{H}_\nu\Big\{f(x)\,;\,y\Big\} = \int_0^\infty \sqrt{x\,y}\,\mathrm{J}_\nu(x\,y)\,f(x)\,dx \tag{1.9}$$

where $\mathrm{J}_\nu(x)$ is the Bessel function of the first kind of order ν. Using the formula (cf. Spanier

& Oldham (1987, p. 306, Eq. 32:13:10))

$$\mathrm{J}_{1/2}(x) = \sqrt{\frac{2}{\pi\, x}}\, \sin(x), \tag{1.10}$$

the definition (1.7) of the Fourier sine transform, and the definition (1.9) of the Hankel transform, we obtain the familiar relationship

$$\mathscr{H}_{1/2}\Big\{ f(x)\,;\, y \Big\} = \sqrt{\frac{2}{\pi}}\, \mathscr{F}_S\Big\{ f(x)\,;\, y \Big\}. \tag{1.11}$$

Similarly, using the formula (cf. Spanier & Oldham (1987, p. 306, Eq. 32:13:11))

$$\mathrm{J}_{-1/2}(x) = \sqrt{\frac{2}{\pi\, x}}\, \cos(x), \tag{1.12}$$

the definition (1.8) of the Fourier cosine transform, and the definition (1.9) of the Hankel transform, we obtain the relationship

$$\mathscr{H}_{-1/2}\Big\{ f(x)\,;\, y \Big\} = \sqrt{\frac{2}{\pi}}\, \mathscr{F}_C\Big\{ f(x)\,;\, y \Big\}. \tag{1.13}$$

The $\mathscr{K}$-transform is defined by

$$\mathscr{K}_\nu\Big\{ f(x)\,;\, y \Big\} = \int_0^\infty \sqrt{x\, y}\, \mathrm{K}_\nu(x\, y)\, f(x)\, dx \tag{1.14}$$

where K_ν is the Bessel function of the second kind of order ν. Using the formula (cf. Spanier & Oldham (1987, p. 239, Eq. 26:13:5))

$$\mathrm{K}_{1/2}(x) = \sqrt{\frac{\pi}{2\, x}}\, \exp(-x), \tag{1.15}$$

the definition (1.2) of the Laplace transform, and the definition (1.14) of the $\mathscr{K}$-transform, we obtain the relationship

$$\mathscr{K}_{1/2}\Big\{ f(x)\,;\, y \Big\} = \sqrt{\frac{\pi}{2}}\, \mathscr{L}\Big\{ f(x)\,;\, y \Big\}, \tag{1.16}$$

which incidentally holds true also when $\mathscr{K}_{1/2}$ is replaced by $\mathscr{K}_{-1/2}$.

In this article, we show that an iteration of the $\mathscr{L}_2$-transform by itself is a constant multiple of the Glasser transform defined by (1.5). Using this iteration identity, we establish a Parseval-Goldstein type theorem relating the $\mathscr{L}_2$-transform and the Glasser transform. The Parseval-Goldstein type theorem established here yields potentially new identities for the various integral transform introduced above. As applications of the resulting identities and theorems, some illustrative examples are also given.

2 The Main Theorem

In the following lemma, we give an iteration identity involving the $\mathscr{L}_2$-transform (1.1) and the Glasser transform (1.5).

Lemma 1 *The identity*

$$\mathscr{L}_2\Big\{\frac{1}{u}\,\mathscr{L}_2\Big\{f(x)\,;\,u\Big\}\,;\,y\Big\}=\frac{\sqrt{\pi}}{2}\,\mathscr{G}\Big\{x\,f(x)\,;\,y\Big\},\tag{2.1}$$

holds true, provided that the integrals involved converge absolutely.

PROOF. Using the definition (1.1) of the $\mathscr{L}_2$-transform, we have

$$\mathscr{L}_2\Big\{\frac{1}{u}\,\mathscr{L}_2\Big\{f(x)\,;\,u\Big\}\,;\,y\Big\}=\int_0^\infty \exp\Big(-y^2\,u^2\Big)\left[\int_0^\infty x\,\exp\Big(-x^2\,u^2\Big)\,f(x)\,dx\right]du.\tag{2.2}$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, and then using the definition (1.1) of the $\mathscr{L}_2$-transform once more, we find from (2.2) that

$$\begin{aligned}\mathscr{L}_2\Big\{\frac{1}{u}\,\mathscr{L}_2\Big\{f(x)\,;\,u\Big\}\,;\,y\Big\}&=\int_0^\infty x\,f(x)\left[\int_0^\infty \exp\Big[\Big(-y^2+x^2\Big)u^2\Big]\,du\right]dx\\&=\int_0^\infty x\,f(x)\,\mathscr{L}_2\Big\{\frac{1}{u}\,;\,\Big(x^2+y^2\Big)^{1/2}\Big\}\,dx.\end{aligned}\tag{2.3}$$

Furthermore, we have

$$\mathscr{L}_2\Big\{\frac{1}{u}\,;\,\Big(x^2+y^2\Big)^{1/2}\Big\}=\frac{\sqrt{\pi}}{2}\,\Big(x^2+y^2\Big)^{-1/2}.\tag{2.4}$$

Now the assertion (2.1) follows from (2.3), (2.4), and the definition (1.5) of the Glasser-transform. □

The Lemma 1 yields some useful corollaries that will be required in our investigation.

Corollary 2 *We have (cf. Glasser (1973, p. 171, (2)))*

$$\mathscr{G}\Big\{x^{\mu-1}\,;\,y\Big\}=2^{-\mu}\,\mathrm{B}\Big(\mu,\frac{1}{2}-\frac{\mu}{2}\Big)\,y^{\mu-1},\quad 0<\Re(\mu)<1,\tag{2.5}$$

where $\mathrm{B}(x,y)$ *is the beta function defined by*

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\quad x>0,y>0,\tag{2.6}$$

and it is related to the gamma function through

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}=\mathrm{B}(y,x). \tag{2.7}$$

PROOF. We set

$$f(x)=x^{\mu-2},\quad 0<\Re(\mu)<1 \tag{2.8}$$

in Lemma 1. Using the relation (1.3) and the known formula Erdelyi et al. (1954, p. 137, Entry (1)), we find that

$$\mathscr{L}_2\Big\{x^{\mu-2}\,;\,u\Big\}=\frac{1}{2}\,\mathscr{L}\Big\{x^{(\mu-2)/2}\,;\,u^2\Big\}=\frac{1}{2}\,\Gamma\Big(\frac{\mu}{2}\Big)\,u^{-\mu}. \tag{2.9}$$

Multiplying the equation (2.9) through by $1/u$ and than applying the $\mathscr{L}_2$-transform, we obtain

$$\mathscr{L}_2\Big\{\frac{1}{u}\,\mathscr{L}_2\Big\{x^{\mu-2}\,;\,u\Big\}\,;\,y\Big\}=\frac{1}{2}\,\Gamma\Big(\frac{\mu}{2}\Big)\,\mathscr{L}_2\Big\{u^{-\mu-1},y\Big\}. \tag{2.10}$$

Using the relation (1.3) and the formula Erdelyi et al. (1954, p. 137, Entry (1)) once more on the right hand side of (2.10), we deduce that

$$\mathscr{L}_2\Big\{\frac{1}{u}\,\mathscr{L}_2\Big\{x^{\mu-2}\,;\,u\Big\}\,;\,y\Big\}=\frac{1}{4}\,\Gamma\Big(\frac{\mu}{2}\Big)\,\Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)\,y^{\mu-1}. \tag{2.11}$$

Utilizing the well-known duplication formula for the gamma function (cf. Spanier & Oldham (1987, p. 414, Eq. (43:5:7)))

$$\Gamma(2\alpha)=\frac{4^{\alpha}}{2\sqrt{\pi}}\,\Gamma(\alpha)\,\Gamma\Big(\frac{1}{2}+\alpha\Big) \tag{2.12}$$

with $\mu=2\alpha$, the relationship (2.7) for the beta function on the right hand side of (2.11) and finally the identity (2.1) of our Lemma 1, we obtain the desired result (2.5). □

Corollary 3 *We have (cf. Glasser (1973, p. 171, (h)))*

$$\mathscr{G}\Big\{x^{\nu+1}\,\mathrm{J}_\nu(z\,x)\,;\,y\Big\}=\sqrt{\frac{2}{\pi\,z}}\,y^{\nu+\frac{1}{2}}\,\mathrm{K}_{\nu+\frac{1}{2}}(z\,y). \tag{2.13}$$

PROOF. We set

$$f(x)=x^{\nu}\,\mathrm{J}_\nu(z\,x),\quad -1<\Re(\nu)<\frac{1}{2} \tag{2.14}$$

in Lemma 1. Using the relation (1.3) and the known formula Erdelyi et al. (1954, p. 185, Entry (30)), we find that

$$\begin{aligned}\mathscr{L}_2\Big\{x^{\nu}\,\mathrm{J}_\nu(z\,x)\,;\,u\Big\}&=\frac{1}{2}\mathscr{L}\Big\{x^{\nu/2}\,\mathrm{J}_\nu\Big(z\,x^{1/2}\Big)\,;\,u^2\Big\}\\&=\frac{1}{2}\,\Big(\frac{z}{2}\Big)^{\nu}\,u^{-2\nu-2}\,\exp\Big(-\frac{z^2}{4u^2}\Big).\end{aligned} \tag{2.15}$$

Multipliying the equation (2.15) through by $1/u$ and than applying the $\mathscr{L}_2$-transform, we obtain

$$\mathscr{L}_2\Big\{\frac{1}{u}\,\mathscr{L}_2\Big\{x^\nu\,\mathrm{J}_\nu(z\,x)\,;\,u\Big\}\,;\,y\Big\} = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \mathscr{L}_2\Big\{u^{-2\nu-3}\,\exp\left(-\frac{z^2}{4u^2}\right)\,;\,y\Big\} \tag{2.16}$$

We evaulate the $\mathscr{L}_2$-transform on the right hand side of (2.16) by using the relation (1.3) and the known formula Erdelyi et al. (1954, p. 146, Entry (29)):

$$\begin{aligned}\mathscr{L}_2\Big\{u^{-2\nu-3}\,\exp\left(-\frac{z^2}{4u^2}\right)\,;\,y\Big\} &= \frac{1}{2}\,\mathscr{L}\Big\{u^{-(2\nu+3)/2}\,\exp\left(-\frac{z^2}{4u}\right)\,;\,y^2\Big\}\\ &= \left(\frac{2y}{z}\right)^{\nu+\frac{1}{2}}\mathrm{K}_{\nu+\frac{1}{2}}(z\,y).\end{aligned} \tag{2.17}$$

Now the assertion (2.13) immediately follows upon substituting the result (2.17) into the equation (2.16) and using the identity (2.1) of our Lemma 1. □

Corollary 4 *We have (cf. Glasser (1973, p. 174, (g)))*

$$\mathscr{G}\Big\{\mathrm{J}_\nu(z\,x)\,;\,y\Big\} = \mathrm{I}_{\nu/2}\Big(\frac{1}{2}\,z\,y\Big)\,\mathrm{K}_{\nu/2}\Big(\frac{1}{2}\,z\,y\Big), \quad \Re(\nu) > -1. \tag{2.18}$$

PROOF. We set

$$f(x) = \frac{1}{x}\,\mathrm{J}_\nu(z\,x), \quad \Re(\nu) > -1 \tag{2.19}$$

in Lemma 1. Using the relation (1.3) and the known formula Erdelyi et al. (1954, p. 185, Entry 29), we find that

$$\mathscr{L}_2\Big\{\frac{1}{x}\,\mathrm{J}_\nu(z\,x)\,;\,u\Big\} = \frac{\sqrt{\pi}}{2u}\exp\left(-\frac{z^2}{8u^2}\right)\mathrm{I}_{\nu/2}\Big(\frac{z^2}{8u^2}\Big). \tag{2.20}$$

Multipliying the equation (2.20) through by $1/u$ and than applying the $\mathscr{L}_2$-transform, we obtain

$$\mathscr{L}_2\Big\{\frac{1}{u}\,\mathcal{L}_2\Big\{\frac{1}{x}\,\mathrm{J}_\nu(z\,x)\,;\,u\Big\}\,;\,y\Big\} = \frac{\sqrt{\pi}}{2}\,\mathscr{L}_2\Big\{\frac{1}{u^2}\,\exp\left(-\frac{z^2}{8u^2}\right)\mathrm{I}_{\nu/2}\Big(\frac{z^2}{8u^2}\Big)\,;\,y\Big\}. \tag{2.21}$$

Now the assertion (2.18) immediately follows upon using the relation (1.3) once more and than utilizing the known formula Prudnikov et al. (1992, p. 325, Entry 10) and (2.1) of our Lemma 1. □

Theorem 5 *If the conditions stated in Lemma 1 are satisfied, then the Parseval-Goldstein type relations*

$$\int_0^\infty \mathscr{L}_2\left\{f(x)\,;\,y\right\}\,\mathscr{L}_2\left\{g(u)\,;\,y\right\}\,dy = \frac{\sqrt{\pi}}{2}\int_0^\infty x\,f(x)\,\mathscr{G}\Big\{u\,g(u)\,;\,x\Big\}\,dx \tag{2.22}$$

$$\int_0^\infty \mathscr{L}_2\left\{f(x)\,;\,y\right\}\,\mathscr{L}_2\left\{g(u)\,;\,y\right\}\,dy = \frac{\sqrt{\pi}}{2}\int_0^\infty u\,g(u)\,\mathscr{G}\Big\{x\,f(x)\,;\,u\Big\}\,du \tag{2.23}$$

and

$$\int_0^\infty x\, f(x)\, \mathscr{G}\Big\{u\, g(u)\,;\, x\Big\}\, dx = \int_0^\infty u\, g(u)\, \mathscr{G}\Big\{x\, f(x)\,;\, u\Big\}\, du \tag{2.24}$$

hold true.

PROOF. We only give the proof of (2.22), as the proof of (2.23) is similar. Identity (2.24) follows from the identities (2.22) and (2.23).

Using the definition (1.1) of the $\mathscr{L}_2$-transform, we have

$$\begin{aligned}&\int_0^\infty \mathscr{L}_2\Big\{f(x)\,;\, y\Big\}\, \mathscr{L}_2\Big\{g(u)\,;\, y\Big\}\, dy \\ &\qquad = \int_0^\infty \mathscr{L}_2\Big\{g(u)\,;\, y\Big\}\, \Big[\int_0^\infty x\, \exp\Big(-x^2\, y^2\Big)\, f(x)\, dx\Big]\, dy.\end{aligned} \tag{2.25}$$

Changing the order of integration (which is permissible by absolute convergence of the integrals involved) and using the definition (1.1) of the $\mathscr{L}_2$-transform once again, we find from (2.25) that

$$\begin{aligned}&\int_0^\infty \mathscr{L}_2\Big\{f(x)\,;\, y\Big\}\, \mathscr{L}_2\Big\{g(u)\,;\, y\Big\}\, dy \\ &\qquad = \int_0^\infty x\, f(x)\, \Big[\int_0^\infty \exp\Big(-x^2\, y^2\Big)\, \mathscr{L}_2\Big\{g(u)\,;\, y\Big\}\, dy\Big]\, dx \\ &\qquad = \int_0^\infty x\, f(x)\, \mathscr{L}_2\Big\{\frac{1}{y}\, \mathscr{L}_2\Big\{g(u)\,;\, y\Big\}\,;\, x\Big\}\, dx.\end{aligned} \tag{2.26}$$

Now the assertion (2.22) easily follows from (2.26) and (2.1) of the Lemma 1. □

Corollary 6 *If the integrals involved converge absolutely and* $0 < \Re(\mu) < 1$, *then we have*

$$\int_0^\infty y^{-\mu}\, \mathscr{L}_2\Big\{f(x)\,;\, y\Big\}\, dy = \frac{1}{2}\, \Gamma\Big(\frac{1}{2} - \frac{\mu}{2}\Big) \int_0^\infty x^{\mu}\, f(x)\, dx, \tag{2.27}$$

$$\int_0^\infty y^{-\mu}\, \mathscr{L}_2\Big\{f(x)\,;\, y\Big\}\, dy = \frac{\sqrt{\pi}}{\Gamma(\mu/2)} \int_0^\infty u^{\mu-1}\, \mathscr{G}\Big\{x\, f(x)\,;\, u\Big\}\, du, \tag{2.28}$$

and

$$\int_0^\infty u^{\mu-1}\, \mathscr{G}\Big\{x\, f(x)\,;\, u\Big\}\, du = \frac{1}{2}\, \mathrm{B}\Big(\frac{\mu}{2}, -\frac{\mu}{2} + \frac{1}{2}\Big) \int_0^\infty x^{\mu}\, f(x)\, dx. \tag{2.29}$$

PROOF. We start with the proof of the assertion (2.27) by setting

$$g(u) = u^{\mu-2} \tag{2.30}$$

in the Theorem 5. Utilizing the formulas (2.5), (2.9) and the identity (2.22) of the Theorem 5, we obtain that

$$\int_0^\infty y^{-\mu}\, \mathscr{L}_2\Big\{f(x)\,;\,y\Big\}\,dy = \frac{\sqrt{\pi}}{2^\mu}\left[\Gamma\Big(\frac{\mu}{2}\Big)\right]^{-1} B\Big(\mu, \frac{1}{2}-\frac{\mu}{2}\Big)\int_0^\infty x^\mu\, f(x)\,dx. \tag{2.31}$$

Using the duplication formula (2.12) for the gamma function with $\mu = 2\alpha$ on the right hand side of (2.31) we deduce the assertion (2.27).

Similarly, the proof of the assertion (2.28) follows upon utilizing (2.30) and (2.9) into the identity (2.23) of our Theorem 5.

Finally, the assertion (2.29) easily follows from the identities (2.27), (2.28) and the relationship (2.7) between the beta function and the gamma function. □

Corollary 7 *If the integrals involved converge absolutely and* $-1 < \Re(\nu) < 1/2$, *then we have*

$$\mathscr{L}_2\Big\{y^{2\nu-1}\,\mathscr{L}_2\Big\{f(x)\,;\,\frac{1}{2y}\Big\}\,;\,z\Big\} = 2^{-\nu-\frac{1}{2}}\,z^{-\nu-1}\,\mathscr{K}_{\nu+\frac{1}{2}}\Big\{x^{\nu+1}\,f(x)\,;\,z\Big\}, \tag{2.32}$$

$$\mathscr{L}_2\Big\{y^{2\nu-1}\,\mathscr{L}_2\Big\{f(x)\,;\,\frac{1}{2y}\Big\}\,;\,z\Big\} = \frac{\sqrt{\pi}}{2^{\nu+1}}\,z^{-\nu-\frac{1}{2}}\,\mathscr{H}_\nu\Big\{u^{\nu+\frac{1}{2}}\,\mathscr{G}\Big\{x\,f(x)\,;\,u\Big\}\,;\,z\Big\}, \tag{2.33}$$

and

$$\mathscr{K}_{\nu+\frac{1}{2}}\Big\{x^{\nu+1}\,f(x)\,;\,z\Big\} = \Big(\frac{\pi\,z}{2}\Big)^{1/2}\,\mathscr{H}_\nu\Big\{u^{\nu+\frac{1}{2}}\,\mathscr{G}\Big\{x\,f(x)\,;\,u\Big\}\,;\,z\Big\}, \tag{2.34}$$

where $\mathscr{H}_\nu\Big\{f(x)\,;\,y\Big\}$ *and* $\mathscr{K}_\nu\Big\{f(x)\,;\,y\Big\}$ *denote the Hankel transform and the* $\mathscr{K}$*-transform as defined by* (1.9) *and* (1.14), *respectively.*

PROOF. We put

$$g(u) = u^\nu\,J_\nu(z\,u) \tag{2.35}$$

in our Theorem 5. Utilizing the identity (2.13) of Corollary 3, the equation (2.15) and the Parseval-Goldstein type relation (2.22) of Theorem 5, we obtain

$$\int_0^\infty \frac{1}{y^{2\nu+2}}\,\exp\Big(-\frac{z^2}{4y^2}\Big)\,\mathscr{L}_2\Big\{f(x)\,;\,y\Big\}\,dy = \Big(\frac{2}{z}\Big)^{\nu+\frac{1}{2}}\int_0^\infty x^{\nu+\frac{3}{2}}\,\mathrm{K}_{\nu+\frac{1}{2}}(z\,x)\,f(x)\,dx. \tag{2.36}$$

Making a simple change of variable in the integral on the left hand side and using the definition (1.14) of the $\mathscr{K}$-transform on the right hand side of (2.36) we obtain the desired identity (2.32).

The assertion (2.33) is obtained similarly using the Parseval-Goldstein relation (2.23) of Theorem 5. The assertion (2.34) immediately follows from the relations (2.22) and (2.23). □

Remark 8 *If we let $\nu = 0$ in our Corollary 7 and than use the formula* (1.10) *and the definition* (1.2) *of the Laplace transform, we obtain*

$$\mathscr{L}_2\left\{\frac{1}{y}\,\mathscr{L}_2\left\{f(x)\,;\,\frac{1}{2y}\right\};\,z\right\} = \frac{\sqrt{\pi}}{2z}\,\mathscr{L}\left\{x\,f(x)\,;\,z\right\} \tag{2.37}$$

$$\mathscr{L}_2\left\{\frac{1}{y}\,\mathscr{L}_2\left\{f(x)\,;\,\frac{1}{2y}\right\};\,z\right\} = \frac{1}{2}\sqrt{\frac{\pi}{z}}\,\mathscr{H}_0\left\{\sqrt{u}\,\mathscr{G}\left\{x\,f(x)\,;\,u\right\};\,z\right\} \tag{2.38}$$

and

$$\mathscr{L}\left\{x\,f(x)\,;\,z\right\} = \sqrt{z}\,\mathscr{H}_0\left\{\sqrt{u}\,\mathscr{G}\left\{x\,f(x)\,;\,u\right\};\,z\right\}. \tag{2.39}$$

Remark 9 *If we let $\nu = -1/2$ in our Corollary 7 and than use the formula* (1.13) *and the definition* (1.8) *of the Fourier cosine transform, we obtain*

$$\mathscr{L}_2\left\{\frac{1}{y^2}\,\mathscr{L}_2\left\{f(x)\,;\,\frac{1}{2y}\right\};\,z\right\} = \frac{1}{\sqrt{z}}\,\mathscr{K}_0\left\{x^{1/2}\,f(x)\,;\,z\right\} \tag{2.40}$$

$$\mathscr{L}_2\left\{\frac{1}{y^2}\,\mathscr{L}_2\left\{f(x)\,;\,\frac{1}{2y}\right\};\,z\right\} = \mathscr{F}_C\left\{\mathscr{G}\left\{x\,f(x)\,;\,u\right\};\,z\right\} \tag{2.41}$$

and

$$\mathscr{K}_0\left\{x^{1/2}\,f(x)\,;\,z\right\} = \sqrt{z}\,\mathscr{F}_C\left\{\mathscr{G}\left\{x\,f(x)\,;\,u\right\};\,z\right\}. \tag{2.42}$$

Remark 10 *If we let $\nu = 1/2$ in our Corollary 7 and than use the formula* (1.11) *and the definition* (1.7) *of the Fourier sine transform, we obtain*

$$\mathscr{L}_2\left\{\mathscr{L}_2\left\{f(x)\,;\,\frac{1}{2y}\right\};\,z\right\} = \frac{1}{2z^{3/2}}\,\mathscr{K}_1\left\{x^{3/2}\,f(x)\,;\,z\right\} \tag{2.43}$$

$$\mathscr{L}_2\left\{\mathscr{L}_2\left\{f(x)\,;\,\frac{1}{2y}\right\};\,z\right\} = \frac{1}{2z}\,\mathscr{F}_S\left\{u\,\mathscr{G}\left\{x\,f(x)\,;\,u\right\};\,z\right\} \tag{2.44}$$

and

$$\mathscr{K}_1\left\{x^{3/2}\,f(x)\,;\,z\right\} = \sqrt{z}\,\mathscr{F}_S\left\{u\,\mathscr{G}\left\{x\,f(x)\,;\,u\right\};\,z\right\}. \tag{2.45}$$

Corollary 11 *If the integrals involved converge absolutely, then we have*

$$\int_0^\infty \frac{1}{y} \exp\left(-\frac{z^2}{8y^2}\right) \mathrm{I}_{\frac{\nu}{2}}\left(-\frac{z^2}{8y^2}\right) \mathscr{L}_2\Big\{f(x)\,;\,y\Big\}\,dy$$
$$= \int_0^\infty x\,f(x)\,\mathrm{I}_{\frac{\nu}{2}}\left(\frac{z\,x}{2}\right) \mathrm{K}_{\frac{\nu}{2}}\left(\frac{z\,x}{2}\right) dx \tag{2.46}$$

$$\int_0^\infty \frac{1}{y} \exp\left(-\frac{z^2}{8y^2}\right) \mathrm{I}_{\frac{\nu}{2}}\left(-\frac{z^2}{8y^2}\right) \mathscr{L}_2\Big\{f(x)\,;\,y\Big\}\,dy$$
$$= z^{-1/2}\,\mathscr{H}_\nu\left\{u^{-1/2}\,\mathscr{G}\Big\{x\,f(x)\,;\,u\Big\}\,;\,z\right\} \tag{2.47}$$

and

$$\int_0^\infty x\,f(x)\,\mathrm{I}_{\frac{\nu}{2}}\left(\frac{z\,x}{2}\right) \mathrm{K}_{\frac{\nu}{2}}\left(\frac{z\,x}{2}\right) dx = z^{-1/2}\,\mathscr{H}_\nu\left\{u^{-1/2}\,\mathscr{G}\Big\{x\,f(x)\,;\,u\Big\}\,;\,z\right\}. \tag{2.48}$$

PROOF. The proof of the Corollary 11 is analogous to the previous Corollary 7. The assertions (2.46), (2.47), and (2.48) are obtained by putting

$$g(u) = \frac{\mathrm{J}_\nu(z\,u)}{u} \tag{2.49}$$

in our Theorem 5 and by using the known formulas Glasser (1973, p. 174, Entry (g)) and Erdelyi et al. (1954, p. 185, Entry (29)). □

The following corollary contains an identity involving $\mathscr{L}_2$-transform, the Glasser transform, the $\mathscr{E}_1$-transform defined by

$$\mathscr{E}_1\Big\{f(x)\,;\,y\Big\} = \int_0^\infty \exp\Big(x\,y\Big)\,\mathrm{E}_1\big(x\,y\big)\,f(x)\,dx, \tag{2.50}$$

introduced in Brown et al. (2007a, p. 1377, Eq. (1.1)), $\mathscr{E}_{2,1}$-transform defined by

$$\mathscr{E}_{2,1}\Big\{f(x)\,;\,y\Big\} = \int_0^\infty x\,\exp\Big(x^2\,y^2\Big)\,\mathrm{E}_1\big(x^2\,y^2\big)\,f(x)\,dx, \tag{2.51}$$

introduced in Brown et al. (2007b, p. 1557, Eq. (1.1)) and the Widder transform defined by

$$\mathscr{P}\Big\{f(x)\,;\,y\Big\} = \int_0^\infty \frac{x\,f(x)}{x^2+y^2}\,dx \tag{2.52}$$

introduced by Widder Widder (1966). The function $\mathrm{E}_1(x)$ is the second member of a family of functions defined by

$$\mathrm{E}_n(x) = \int_1^\infty \frac{\exp(-x\,t)}{t^n} \qquad n = 0, 1, \ldots\,. \tag{2.53}$$

The functions $\mathrm{E}_n(x)$ were introduced by Schlömilch. The function $\mathrm{E}_1(x)$ in the definitions (2.50) and (2.51) of the $\mathscr{E}_{2,1}$ is related in a simple way to exponential integral function:

$$\mathrm{E}_1(x) = -\mathrm{Ei}(-x), \tag{2.54}$$

where the exponential integral function is defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{\exp(t)}{t}\, dt. \tag{2.55}$$

Corollary 12 *If the integrals involved converge absolutely, then we have*

$$\mathscr{E}_{2,1}\left\{\frac{1}{y}\,\mathscr{L}_2\Big\{f(x)\,;\,y\Big\}\,;\,z\right\} = \sqrt{\pi}\,\mathscr{P}\Big\{\mathscr{G}\Big\{x\,f(x)\,;\,u\Big\}\,;\,z\Big\}. \tag{2.56}$$

PROOF. The assertion (2.56) immediately follows upon putting

$$g(u) = \frac{1}{u^2+z^2} \tag{2.57}$$

the identity (2.23) of our Theorem 5 and by using the known formula Erdelyi et al. (1954, p. 185, Entry (29)).

3 Illustrative Examples

An interesting illustration for the identity (1) asserted by Lemma 1 is contained in the following example.

Example 13 *Suppose that* $|z| > |y|$. *Then*

$$\mathscr{L}_2\left\{\frac{1}{u}\,\exp\Big(z^2\,u^2\Big)\,E_1\Big(z^2\,u^2\Big)\,;\,y\right\} = \sqrt{\pi}\,\frac{\pi - 2\,\arcsin(y/z)}{\sqrt{z^2-y^2}}, \tag{3.1}$$

$$\mathscr{L}\left\{\frac{1}{\sqrt{u}}\,\exp\Big(z\,u\Big)\,E_1\Big(z\,u\Big)\,;\,y\right\} = \sqrt{\pi}\,\frac{\pi - 2\,\arcsin\Big(\sqrt{y/z}\Big)}{\sqrt{z-y}}, \tag{3.2}$$

$$\mathscr{E}_{2,1}\left\{\frac{1}{u}\,\exp\Big(-y^2\,u^2\Big)\,;\,z\right\} = \sqrt{\pi}\,\frac{\pi - 2\,\arcsin(y/z)}{\sqrt{z^2-y^2}}, \tag{3.3}$$

and

$$\mathscr{E}_1\left\{\frac{1}{\sqrt{u}}\,\exp\Big(-y\,u\Big)\,;\,z\right\} = \sqrt{\pi}\,\frac{\pi - 2\,\arcsin\Big(\sqrt{y/z}\Big)}{\sqrt{z-y}}. \tag{3.4}$$

PROOF. We put

$$f(x) = \frac{1}{x^2 + z^2}\cdot \tag{3.5}$$

Using the known result Apelblat (1983, p. 10, Entry (47)) we find that

$$\mathscr{G}\left\{\frac{x}{x^2+z^2}\,;\,y\right\} = \frac{\pi - 2\,\arcsin(y/z)}{2\sqrt{z^2-y^2}}\cdot \tag{3.6}$$

Using the relationship (1.3) between the Laplace transform and the $\mathscr{L}_2$-transform and the known formula Prudnikov et al. (1992, p. 17, Entry (5)), we obtain

$$\mathscr{L}_2\left\{\frac{1}{x^2+z^2}\,;\,u\right\} = \frac{1}{2}\,\mathscr{L}\left\{\frac{1}{x+z^2}\,;\,u^2\right\} = \frac{1}{2}\,\exp\left(z^2\,x^2\right)\mathrm{E}_1\left(z^2\,x^2\right)\cdot \tag{3.7}$$

Substituting the results (3.6) and (3.7) into the identity (2.1) of our Lemma 1, we obtain the asserted formula (3.1).

From the relationship (1.3) we deduce the assertion (3.2). The assertions (3.3) and (3.4) follow from the definitions (2.50) and (2.51) of the $\mathscr{E}_1$-transform and $\mathscr{E}_{2,1}$-transform, respectively. □

The following illustration involves the error function defined by

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-x^2\right)dx, \tag{3.8}$$

and the Dawson integral defined by

$$\mathrm{daw}(x) = \int_0^x \exp\left(t^2 - x^2\right)dt. \tag{3.9}$$

The Dawson integral and the error function are related via the identity

$$\mathrm{daw}(x) = \frac{-i\sqrt{\pi}}{2}\,\exp\left(-x^2\right)\mathrm{Erf}(i\,x) \tag{3.10}$$

(cf. Spanier & Oldham (1987, p. 405, Eq. 42:0:1))).

Example 14 *We show*

$$\mathscr{L}_2\left\{\frac{1}{u^2}\,\exp\left(-\frac{z^2}{4u^2}\right)\mathrm{Erf}\left(i\,\frac{z}{2u}\right);\,y\right\} = \frac{\pi\,i}{2}\left[\mathrm{I}_0(z\,y) - \mathbf{L}_0(z\,y)\right] \tag{3.11}$$

and

$$\mathscr{L}_2\left\{\frac{1}{u^2}\,\mathrm{daw}\left(\frac{z}{2u}\right);\,y\right\} = \frac{\pi^{3/2}}{4}\left[\mathrm{I}_0(z\,y) - \mathbf{L}_0(z\,y)\right], \tag{3.12}$$

where $\mathrm{I}_0(x)$ *denotes the modified Bessel function of the first kind of order zero and* $\mathrm{L}_0(x)$ *denotes the modified Struve function of order zero.*

PROOF. We put

$$f(x) = \frac{\sin(z\,x)}{x}\cdot \tag{3.13}$$

Using the relationship (1.3) and the known formula Erdelyi et al. (1954, p. 154, Entry (36)), we have

$$\begin{aligned}\mathscr{L}_2\Big\{x^{-1}\,\sin(z\,x)\,;\,u\Big\} &= \frac{1}{2}\,\mathscr{L}\Big\{x^{-1/2}\,\sin(z\,x^{1/2})\,;\,u^2\Big\}\\ &= -\frac{i\sqrt{\pi}}{2u}\,\exp\Big(-\frac{z^2}{4u^2}\Big)\,\mathrm{Erf}\Big(i\,\frac{z}{2u}\Big)\cdot\end{aligned} \tag{3.14}$$

Multiplying both sides of (3.14) by $1/u$ and then applying the $\mathscr{L}_2$-transform, we find that

$$\mathscr{L}_2\Big\{\frac{1}{u}\,\mathscr{L}_2\Big\{x^{-1}\,\sin(z\,x)\,;\,u\Big\}\,;\,y\Big\} = -\frac{i\sqrt{\pi}}{2}\,\mathscr{L}_2\Big\{\frac{1}{u^2}\,\exp\Big(-\frac{z^2}{4u^2}\Big)\,\mathrm{Erf}\Big(i\,\frac{z}{2u}\Big)\,;\,y\Big\}\cdot \tag{3.15}$$

From the known formula Glasser (1973, p. 174, Entry (a)) we have

$$\mathscr{G}\Big\{\sin(z\,x)\,;\,y\Big\} = \frac{\pi}{2}\,\Big[\mathrm{I}_0(z\,y) - \mathbf{L}_0(z\,y)\Big]\cdot \tag{3.16}$$

Substituting the formulas (3.15) and (3.16) into the identity (1) of our Lemma 1, we obtain the desired result (3.11).

From the relationship (3.10) and the formula (3.11) we deduce, the assertion (3.12). □

Remark 15 *Using the relationship* (1.3) *and setting* $z = 2a^{1/2}$ *we can restate the formulas* (3.11) *and* (3.12) *as*

$$\mathscr{L}\Big\{\frac{1}{u}\,\exp\Big(-\frac{a}{u}\Big)\,\mathrm{Erf}\Big(i\sqrt{\frac{a}{u}}\Big)\,;\,y\Big\} = \pi\,i\,\Big[\mathrm{I}_0\Big(2\sqrt{a\,y}\Big) - \mathbf{L}_0\Big(2\sqrt{a\,y}\Big)\Big] \tag{3.17}$$

and

$$\mathscr{L}\Big\{\frac{1}{u}\,\mathrm{daw}\Big(\sqrt{\frac{a}{u}}\Big)\,;\,y\Big\} = \pi\,\Big[\mathrm{I}_0\Big(2\sqrt{a\,y}\Big) - \mathbf{L}_0\Big(2\sqrt{a\,y}\Big)\Big]. \tag{3.18}$$

Example 16 *Suppose that* $\Re(z) > 0$ *and* $\max\{0, -2\,\Re(\nu)\} < \Re(\mu) < 1$. *Then*

$$\int_0^\infty y^{-\mu-1}\,\exp\Big(-\frac{z^2}{2y^2}\Big)\,\mathrm{I}_\nu\Big(\frac{z^2}{2y^2}\Big)\,dy = \frac{\Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)\,\Gamma\Big(\nu+\frac{\mu}{2}\Big)}{2\sqrt{\pi}\,z^\mu\,\Gamma\Big(\nu-\frac{\mu}{2}+1\Big)} \tag{3.19}$$

and

$$\int_0^\infty u^{\mu-1}\,\mathrm{I}_\nu(z\,y)\,\mathrm{K}_\nu(z\,y)\,dy = \frac{\Gamma\Big(\frac{\mu}{2}\Big)\,\Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)\,\Gamma\Big(\nu+\frac{\mu}{2}\Big)}{4\sqrt{\pi}\,z^\mu\,\Gamma\Big(\nu-\frac{\mu}{2}+1\Big)} \tag{3.20}$$

(cf. Kahramaner et al. (1995, p. 13, Eq. (4.8))).

PROOF. Putting

$$f(x) = \frac{\mathrm{J}_{2\nu}(2zx)}{x} \tag{3.21}$$

into the identity (2.27) of Corollary 6, we obtain

$$\int_0^\infty y^{-\mu}\, \mathscr{L}_2\left\{\frac{\mathrm{J}_{2\nu}(2zx)}{x}\,;\, y\right\} dy = \frac{1}{2}\,\Gamma\left(\frac{1}{2} - \frac{\mu}{2}\right) \int_0^\infty x^{\mu-1}\, \mathrm{J}_{2\nu}(2zx)\, dx, \tag{3.22}$$

Utilizing the formulas (2.20) and Erdelyi et al. (1954, p.326, Entry (1)) together with (3.22) we obtain the assertion (3.19).

Similarly, using the formula (2.18) and Erdelyi et al. (1954, p.326, Entry (1)) together with (2.29) we deduce the second assertion (3.20) of our Example 16.

We conclude this investigation by remarking that many other infinite integrals can be evaluated in this manner by applying the above Lemma, the above Theorem, and their various corollaries and consequences considered here.

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