DERIVING A FORMULA FOR SUMS OF POWERS OF INTEGERS

Thomas J Pfaff
Department of Math & CS
Ithaca College
Ithaca, NY 14850
tpfaff@ithaca.edu

Students typically encounter the formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

in calculus or discrete math courses. At some point they may prove these results by induction. The question, of course, is where did these formulas come from in the first place? Here, we will provide a simple approach for deriving formulas for $S_k(n) = \sum_{i=1}^{n} i^k$ and relevant facts about $S_k(n)$, along with a closed form for $S_k(n)$ by way of the Stirling numbers of the second kind.

Our motivation comes from a slightly generalized problem from [8] that asks the following:

> Consider a group of $n+1$ people. If everyone shakes hands with everyone else, how many handshakes take place?

This problem can be solved in two different ways. Using combinations, we see that there are $\binom{n+1}{2}$ handshakes that take place. Another solution is to label the people $p_{n+1}, p_n, \ldots, p_1$ and then have person p_{i+1} shake the hands of everyone with an index less than $i+1$. Since p_{i+1} will shake i hands, we have $\sum_{i=1}^{n} i$ handshakes that take place. Equating the two solutions, yields the well known formula

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

Formulas for $S_k(n)$ are certainly well known and can be obtained from Mathematica or found in [9]. There have also been a number of papers addressing such sums, including [1], [3], [7], although only a few providing a combinatorial approach to finding $S_k(n)$ such as

[2], [6]. A nice recursion method for calculating $S_k(n)$ can be found in [5]. We now turn to generalizing this method to obtain a formula for $S_k(n) = \sum_{i=1}^{n} i^k$.

1. The Generalized Formula

Before moving to the general result we will provide an example for finding $S_3(n)$. Instead of talking about handshakes, we will talk about meetings of various sizes. For the $k = 3$ case we want $n + 1$ people to meet in all possible groups of size four 6 times, groups of size three 6 times, and by groups of size two (the handshake case) 1 time. The seemingly odd choice of 6, 6 and 1 will be explained in a moment. As in the introduction, we label the people $p_{n+1}, p_n, \ldots, p_1$ and then have person p_{i+1} go through and meet with everyone with an index less than $i + 1$, in the appropriate meeting sizes. We count the group meetings of size 4 for p_{i+1} with people of index less than $i + 1$, by making all possible groups of size 3 with $p_1, \ldots, p_i$, which is $\binom{i}{3}$, and then add p_{i+1} to the group. Similarly, we count the groups of size 3 and 2 for p_{i+1} to get $\binom{i}{2}$ and $\binom{i}{1}$ respectively. Thus, accounting for the requested multiplicity of 6, 6, and 1, person p_{i+1} will have

$$6\binom{i}{3} + 6\binom{i}{2} + \binom{i}{1} = i^3 \tag{1}$$

encounters to make with the people of index less than $i + 1$.

The repeated meetings of 6, 6, and 1 are chosen recursively so that the number of encounters person p_{i+1} has in (1) equals i^3. To see that this can be done note that

$$\begin{aligned} \binom{i}{3} &= \frac{i^3 - 3i^2 + 2i}{6} \\ \binom{i}{2} &= \frac{i^2 - i}{2} \\ \binom{i}{1} &= \frac{i}{1}, \end{aligned}$$

with the first term containing the only i^3, and the first two terms having the only i^2. Hence the first 6 is chosen so that $6\binom{i}{3}$ leaves us with an i^3. The second 6 yields a $3i^2$ from $6\binom{i}{2}$ to cancel the $-3i^2$ from $6\binom{i}{3}$. Finally, $6\binom{i}{3} + 6\binom{i}{2} = i^3 - i$, and so we need only a 1 for the last term to cancel the $-i$.

On the other hand, the number of meetings can be counted simply by

$$6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2}. \tag{2}$$

Hence summing (1) over people p_2 through p_{n+1}, note that p_1 has no encounters left to make with anyone else, and equating it to (2) yields

$$\sum_{i=1}^{n} i^3 = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2}.$$

To obtain a formula for $S_k(n)$, we again label the people $p_{n+1}, p_n, \ldots, p_1$. Now each person will go through and meet in groups of size $j+1$, c_j^k times for $j = 1, \ldots, k$ with everyone with an index less than $i+1$. The c_j^k are chosen so that when counting the number of meetings for person p_{i+1} we get that

$$\sum_{j=1}^{k} c_j^k \binom{i}{j} = i^k. \tag{3}$$

To see that the c_j^k exist and are independent of i note that the left hand side is a polynomial in i. When $i > k$ the c_j^k can be easily chosen recursively as in the example. The only combination that will produce i^k is $\binom{i}{k}$, and so $c_k^k = k!$ so that the coefficient of i^k will be one. The only other combination that will produce an i^{k-1} term will be $\binom{i}{k-1}$, and so the c_{k-1}^k will be chosen so that the coefficient of the i^{k-1} terms is 0. In fact, with a bit of counting we get that $c_{k-1}^k = k!(k-1)/2$. We thus proceed in this fashion. Note that the choices of c_j^k were independent of i. But now both sides of (3) are polynomials of degree k that agree on $k+1$ distinct points, in fact for all $i > k$, and so the polynomials are identical. Hence the c_j^k exist, are independent of i, and (3) holds for all positive integers i.

Since these computations are all the same for any $i > k$, we have two polynomials of degree k that agree on $k+1$ distinct points and hence the polynomials are identical.

On the other hand, the total number of meetings can be found by

$$\sum_{j=1}^{k} c_j^k \binom{n+1}{j+1}. \tag{4}$$

Summing (3) from $i = 1$ to n and setting it equal to (4) proves

Theorem 1.1. *For any positive integers k and n*

$$S_k(n) = \sum_{j=1}^{k} c_j^k \binom{n+1}{j+1}, \tag{5}$$

where $c_1^k \dots c_k^k$ satisfy, for all i,

$$\sum_{j=1}^{k} c_j^k \binom{i}{j} = i^k. \tag{6}$$

The result in (5) is also given in [6], but here we add the relationship among the coefficients c_j^k. We can now make the following observations about $S_k(n)$, some of which are in [6]:

i. The function $S_k(n)$ is a polynomial in n of degree $k+1$, with rational coefficients. As noted in [1], a proof of this was given by Pascal in 1654 by showing that

$$\begin{aligned}(n+1)^{k+1} - 1 &= \sum_{i=1}^{n} \left[(i+1)^{k+1} - i^{k+1}\right] \\ &= \sum_{i=1}^{n} \sum_{j=0}^{k} \binom{k+1}{j} i^j \\ &= \sum_{j=0}^{k} \binom{k+1}{j} S_n(j).\end{aligned}$$

One now uses induction to prove that $S_k(n)$ is a polynomial in n of degree $k+1$.

ii. From (5) we see that $S_k(n)$ always has a factor of $n(n+1)$.

iii. Since $c_k^k = k!$, the leading coefficient of $S_k(n)$ is $1/(k+1)$. Moreover, since $c_{k-1}^k = k!(k-1)/2$, the coefficient of the i^k term is always $1/2$.

iv. Using (6) with $i = 1$ give us that $c_1^k = 1$. By the same process with $i = 1, 2, 3$, and 4 we get that, $c_2^k = 2^k - 2$, $c_3^k = 3^k - 3(2^k) + 3$, and $c_4^k = 4^k - 4(3^k) + 6(2^k) - 4$. From this it is not hard to calculate that

$$\begin{aligned}S_1(n) &= \frac{n^2}{2} + \frac{n}{2} \\ S_2(n) &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\ S_3(n) &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \\ S_4(n) &= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.\end{aligned}$$

From iv above, one might wonder if the $c_j^k \geq 0$. They are, but it isn't clear how to see this based on their construction here. In fact, more general results exits to find the partial sums of polynomials, see section 8.2 in [4], by way of difference sequences. We have only provided partial sum formulas for polynomials of the form n^k. It turns out that from difference sequences there is an interesting way to calculate the c_j^k, again see [4].

For example to find c_i^3 for $i = 1, 2,$ and 3, start the first row of a difference table by evaluating i^3 for $i = 0$ to 4 to make the following difference table, where each integer is the difference of the pair above:

$$\begin{array}{ccccccccc} 0 & & 1 & & 8 & & 27 & & 64 \\ & 1 & & 7 & & 19 & & 37 & \\ & & 6 & & 12 & & 18 & & \\ & & & 6 & & 6 & & & \\ & & & & 0 & & & & \end{array}$$

We now find that c_i^3 is the first element in row i, where the first row is row 0. Since each row is increasing we can see that the $c_i^k \geq 0$, though this is not obvious by our choice of them.

2. A Closed Form for $S_k(n)$ and Some Identities

The c_j^k can be put in closed form by way of Stirling numbers of the second kind (yes, there are Stirling numbers of the first kind). Let $\left\{ {k \atop j} \right\}$ be the Stirling number of the second kind, also known as the Stirling subset number, which counts the number of ways that k distinguishable balls can be put into j indistinguishable cells with no cell empty.

We should point out the three "triangles" that we encounter here. Of course, the binomial coefficients can be found by way of Pascal's triangle, and we have the difference triangle for the c_j^k. We can find $\left\{ {k \atop j} \right\}$ in a similar way. The table below contains the Stirling numbers of the second kind where each entry is found taking the number directly above, multiplying it by the column number, and adding the number diagonally up to the left of the given entry.

It turns out that the $\left\{ {k \atop j} \right\}$ are coefficients of a type of polynomial. Let $x^{(i)} = x(x-1)\ldots(x-i+1)$, which is called the factorial polynomial with $x^0 = 1$. From [9], we get that, for $k \neq 0$,

$$i^k = \sum_{j=1}^{k} \left\{ {k \atop j} \right\} i^{(j)} \tag{7}$$

k \ j	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	1	3	1				
4	0	1	7	6	1			
5	0	1	15	25	10	1		
6	0	1	31	90	65	15	1	
7	0	1	63	301	350	140	21	1

Now since the c_j^k are chosen to satisfy (6), each c_j^k will have a factor of $j!$ to cancel the $j!$ that appears in the denominator of $\binom{i}{j}$. It thus makes sense to set $c_j = j!b_j^k$, where b_j^k is an integer. Finally, in a few quick steps we can get a closed form for $S_k(n)$. From (6) we have that

$$i^k = \sum_{j=1}^{k} c_j^k \binom{i}{j} = \sum_{j=1}^{k} j!\, b_j^k \binom{i}{j} = \sum_{j=1}^{k} b_j^k i^{(j)},$$

and so, from (7),

$$b_j^k = \left\{ {k \atop j} \right\}.$$

Again from [9] we learn that

$$\left\{ {k \atop j} \right\} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^k.$$

Thus,

$$c_j^k = j! \left\{ {k \atop j} \right\} = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^k,$$

which provides the closed form for $S_k(n)$,

$$S_k(n) = \sum_{j=1}^{k} \sum_{i=0}^{j} (-1)^{j-i}\, i^k \binom{j}{i} \binom{n+1}{j+1}. \tag{8}$$

This approach can be used to obtain results for other sums. For instance to show that, for $n > k > 1$,

$$\sum_{i=1}^{n} i(i-1)\dots(i-k+2) = (k-1)!\binom{n+1}{k} \tag{9}$$

have all $n+1$ people meet in groups of size k repeated $(k-1)!$ times. There are a total of $(k-1)!\binom{n+1}{k}$ meetings. On the other hand, person p_{i+1} will meet

$$(k-1)!\binom{i}{k-1} = i(i-1)\dots(i-k+2)$$

times with individuals with index less than $i+1$. Summing this yields (9). In other words, we have simply shown that

$$\sum_{i=1}^{n} \binom{i}{k-1} = \binom{n+1}{k},$$

which follows easily from Pascal's triangle, is often referred to as the Hockey Stick Lemma, and can be given a more straightforward combinatorial interpretation.

We noted in the introduction that there are few combinatorial proofs for formulas for sums of powers of integers. In fact, currently there isn't a combinatorial proof to prove (8) directly. It seems that one should exits.

References

[1] A. Beardon, Sums of Powers of Integers, *Amer. Math. Monthly* 103 (1996), 201-213.

[2] A. Benjamin and M. Orrison, Two Quick Combinatorial Proofs of $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$, *College Mathematics Journal* 33 (2002), 406-408.

[3] D. Bloom, An Old Algorithm for the Sums of Integer Powers, *Mathematics Magazine* 66 (1993), 304-305.

[4] R. A. Brualdi, *Introductory Combinatorics,* 3rd ed. Prentice-Hall, Upper Saddle River, New Jersey 1999.

[5] A. Saleh-Jahromi and J. Doucet, The Sum of the kth Powers of the First N Positive Integers, *The* Π*ME Journal*, vol. 11, no. 5, 259-261. 2001.

[6] G. Mackiw, A Combinatorial Approach to Sums of Integer Powers, *Mathematics Magazine* 73 (2000), 44-46.

[7] R. Owens, Sums of Powers of Integers, *Mathematics Magazine* 65 (1992), 38-40.

[8] S.M. Ross, *A first Course in Probability,* 5th ed. Prentice-Hall, Upper Saddle River, New Jersey 1997.

[9] D. Zwillinger (editor), *Standard Mathematical Tables and Formula,* 31st ed. CRC Press, Boca Raton, Florida 2003.