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A new approach to Macaulay posets

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Abstract

We develop a new approach for establishing the Macaulayness of posets representable as cartesian powers of other posets. This approach is based on a problem of constructing an ideal of maximum rank in a poset. Using the relations between the maximum rank ideal problem and the edge-isoperimetric problem on graphs we demonstrate an application of our approach to specification of all posets with a special Macaulay order. We also present a new general construction for additive Macaulay posets and introduce several new families of Macaulay posets.

1. Introduction

Let $(P, \leqslant_P)$ be a poset with a partial order $\leqslant_P$. We say P is *ranked* if there exists a function $r_P : P \mapsto \mathbb{N}$ such that $r_P(z) = 0$ for some minimal element $z \in P$, and $r_P(x) = r_P(y) - 1$ whenever $x <_P y$ and there is no $z \in P$ with $x <_P z <_P y$. Note that we do not suppose that $r_P(x) = 0$ for all minimal elements of P. We will omit the subscript in r_P if it is clear which poset is considered. For $i \geqslant 0$ denote by P_i the set of all elements of P of rank i and let $r(P) = \max_{x \in P} r(x)$. Now, for $0 < i \leqslant r(P)$ and $A \subseteq P_i$ define the *shadow* of A as

$$\Delta(A) = \{x \in P_{i-1} \mid x \leqslant_P y \text{ for some } y \in A\}.$$

We put $\Delta(A) = \emptyset$ for any $A \subseteq P_0$. For fixed $i > 0$ and m, $1 \leqslant m \leqslant |P_i|$, the shadow minimization problem (SMP) consists of finding a set $A \subseteq P_i$ such that $|A| = m$ and

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$|\Delta(A)| \leqslant |\Delta(B)|$ for any $B \subseteq P_i$, $|B| = m$. Sets that are solutions to the SMP are called *optimal*. SMP is one of fundamental problems in combinatorics and has numerous applications. Examples of such problems include computing the number of perfect matchings in bipartite graphs, the number of monotone Boolean functions, percolation problems, network reliability, and extremal problems in combinatorial topology. For details and more examples the reader is referred to a book of Engel [11] and a survey [6].

It turns out that for many posets the solutions to the SMP are nested in the sense that there exists a total order of the elements of P_i, such that any initial segment of this order is an optimal set. Moreover, the nested optimal subsets in those posets satisfy many important properties, which leads to the notion of a *Macaulay poset*. A poset $(P, \leqslant_P)$ is called Macaulay if there exists a total order $\mathcal{M}$ (called the *Macaulay order*) on P such that for any initial segment S of this order and for any $i > 0$ the following two conditions are satisfied:

Nestedness. the set $S \cap P_i$ is optimal.

Continuity. $\Delta(S \cap P_i) = S' \cap P_{i-1}$ for some initial segment S' of order $\mathcal{M}$.

We consider the SMP on posets that are representable as the *cartesian product* of other posets. Given two posets $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$, their cartesian product is a poset $(P \times Q, \leqslant_{P \times Q})$, such that $(x, y) \leqslant_{P \times Q} (x', y')$ if and only if $x \leqslant_P x'$ and $y \leqslant_Q y'$. Since this operation is associative, the *cartesian powers* of a poset are well defined. We denote the nth cartesian power of $(P, \leqslant_P)$ by P^n. It is easily shown that the cartesian product of ranked posets is a ranked poset. In particular, for $\mathbf{x} = (x_1, \ldots, x_n) \in P^n$ one has

$$r_{P^n}(\mathbf{x}) = \sum_{i=1}^{n} r_P(x_i). \tag{1}$$

The SMP for the cartesian powers of a Macaulay poset has been intensively studied in the literature. The examples include the famous Kruskal–Katona and Clements–Linström theorems that establish the Macaulayness of the Boolean lattice and lattice of multisets, respectively. Further examples include the star poset [11] and the spider poset [5] (see [6] for more examples). The Macaulay orders for these posets vary from the *lexicographic order* to a rather complicated order for the spider poset. We say that $(x_1, \ldots, x_n) \in P^n$ precedes $(y_1, \ldots, y_n) \in P^n$ in the lexicographic order if $x_1 = y_1, \ldots, x_{i-1} = y_{i-1}$ and $x_i <_P y_i$ for some i, $1 \leqslant i \leqslant n$.

Concerning the lexicographic order, a *local–global principle* [7] tells us that if this order is Macaulay for P^2 then, under certain natural assumptions on P, it is Macaulay for P^n for any $n \geqslant 3$. This result is an extension of the local–global principle discovered by Ahlswede and Cai [1] with respect to the *edge-isoperimetric problem* (EIP) on graphs. The EIP for a given graph $G = (V_G, E_G)$ and integer m consists of finding an induced subgraph $S = (V_S, E_S)$ of G such that $|V_S| = m$ and $|E_S|$ is maximum among all induced subgraphs with the same number of vertices (see [2] for a survey). It seems to be very challenging to specify all posets for which the lexicographic order is Macaulay. Moreover, no local–global principle for the SMP is known for other orders.

We extend the concept of nestedness in the SMP to other extremal problems defined on subsets of elements of a graph or a poset of order p. A series of solutions $A_1, A_2, \ldots, A_p$, $|A_i| = i$, to a specific problem is called *nested* if $A_i \subset A_{i+1}$ for $i = 1, \ldots, p-1$. In other words, there exist a total order defined on the element set of the graph or poset, such that any initial segment of this order provides a solution to the considered problem.

The SMP is closely related to the maximum rank ideal (MRI) problem. A set $I \subseteq P$ of a ranked poset $(P, \leqslant_P)$ is called *ideal* if for any $x \in I$ one has $y \in I$ whenever $y \leqslant_P x$. For an ideal I we define its rank by $R(I) = \sum_{x \in I} r(x)$. The MRI problem consists of finding for a given m, $1 \leqslant m \leqslant |P|$, an ideal $I \subseteq P$, such that $|I| = m$ and $R(I) \geqslant R(I')$ for any ideal $I' \subseteq P$ with $|I'| = m$. The MRI problem is a special case of the maximum weight ideal problem studied in [12], where instead of $r(x)$ an arbitrary weight function is used in the definition of $R(I)$. The MRI problem is, in a sense, easier than the SMP due to the specific properties of the weight function that are not valid for the shadow function. It is known that the MRI problem on a Macaulay poset does have nested solutions. However, the corresponding MRI-order can be different from the Macaulay order, in general. Moreover, the existence of an MRI-order does not imply the poset is Macaulay [3].

The MRI problem, in turn, is closely related to the edge-isoperimetric problem on graphs. An algorithm presented in [3] constructs for any graph G a *representing* Macaulay poset P. This poset admits nested solutions in the MRI problem such that if S is a solution to the EIP on G and $I \subseteq P$ is a solution to the MRI on P, then $|E_S| = R(I)$ whenever $|V_S| = |I|$. Thus, EIP on G is equivalent to the MRI problem on P. It turns out that P^n is a representing poset for the nth cartesian power of G for any $n \geqslant 1$. Since the nestedness in the MRI problem on a poset is a necessary condition for the poset to be Macaulay and the MRI problem is easier to solve, it makes sense to check the poset for the nestedness in the MRI before verifying its Macaulayness. It seems that the Macaulayness of the cartesian powers P^n is valid due not only to the structural properties of the basic poset P, but also to the properties of the corresponding Macaulay orders. This moves the focus in the analysis of Macaulay posets to the analysis of orders.

In the light of the results presented above, the MRI problem is closely related to the EIP, where very powerful methods have been developed. A new idea that we explore in our paper is to adapt some EIP methods to the MRI problem and approach the SMP through the MRI. We demonstrate this idea for a specific order $\mathscr{L}^n$ defined on P^n for $n \geqslant 1$. We show that under certain conditions on P^2, P^3, and P^4 the order $\mathscr{L}^n$ provides solutions to *both* the SMP and the MRI problem on P^n for any $n \geqslant 2$. Via the analysis of a simpler problem, the MRI, we narrow the class of posets for which the order $\mathscr{L}^n$ can be an MRI-order, and thus a Macaulay order. After that each poset in this class is checked for the Macaulayness. With our new approach we not only establish a local–global principle for the order $\mathscr{L}^n$ with respect to the SMP, but also characterize all posets for which cartesian powers with this order are Macaulay. As a byproduct we also specify all graphs for which $\mathscr{L}^n$ provides the nestedness for the EIP.

Presently a limited number of orders are involved in the EIP/MRI business. The lexicographic order and the star order were the only examples for a long time. Recently, a new order for the cartesian powers of the Petersen graph was introduced [4] and its restriction to the powers of cycles of length 5 was studied [9]. The last order admits a natural generalization for larger graphs (and posets), and we call the generalized order the *zigzag order* (see Section 3 for a precise definition of $\mathscr{Z}^n$). The zigzag order in two dimensions is rather close to the well-studied lexicographic order. It is known that the lexicographic order provides the nestedness for the EIP on many graphs [2]. It is interesting that even a slight modification of this order leads, as we show, to a significant narrowing of the class of graphs it is applicable to.

We present two new series of posets for which cartesian powers with the zigzag order are Macaulay. One of these posets is the representing poset of a 5-cycle. The Macaulayness of this poset implies a result of [9] concerning the EIP for the cartesian powers of 5-cycles. The other poset is the so-called N-poset. To prove its Macaulayness we developed a new general technique which is applicable to many other posets.

The paper is organized as follows. In Section 2 we present some auxiliary results needed for the proofs in Sections 3–5. Section 3 is devoted to the definition of the zigzag order and to proving Theorem 1, which provides a characterization of all posets for whose cartesian powers the zigzag order is an MRI-order. It turns out that there are just two interesting posets with this property: the N-poset and the diamond poset M_3 shown in Fig. 4(b) and Fig. 2(a), respectively. Since the zigzag order is only a slight modification of the lexicographic order it is surprising that there are only two posets with this property whereas there are many for the lexicographic order. Also in the section are Lemmas 5–9, which various consistent Macaulay orders share many common properties with. The proofs of theses properties are interesting as they follow a common pattern which may possibly be extended to a wider class of orders. In Section 4 we prove the Macaulayness of the diamond poset in Theorem 2. This theorem implies one of the main results of [9]: a solution of an edge-isoperimetric problem on the cartesian product of 5-cycles. In Section 5 we present a new general construction for Macaulay posets formulated in Theorem 3. This is a kind of new product theorem for Macaulay posets, and we apply it to establishing the Macaulayness of the powers of the N-poset. This is the first non-trivial Macaulay poset with more than one minimum and maximum elements, whose all cartesian posets are Macaulay. Concluding remarks in Section 6 complete the paper.

2. Some auxiliary results

Let P be a ranked poset with a total order $\mathcal{O}^1$ defined on its element set. Denote $p = |P| - 1$ and assume $P = \{0, 1, \ldots, p\}$, where the elements are listed in the order $\mathcal{O}^1$. Now the elements of P^n can be considered as n-dimensional vectors with integer entries from the set P. Furthermore, suppose for any $n \geqslant 2$ a total order $\mathcal{O}^n$ is defined on the set P^n. We say that the order $\mathcal{O}^n$ is *consistent* if for any two elements

$\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ of P^n and any $i = 1, 2, \ldots, n$ whenever $x_i = y_i$ one has: $(x_1, \ldots, x_i, \ldots, x_n) <_{\mathcal{O}^n} (y_1, \ldots, y_i, \ldots, y_n)$ if and only if $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) <_{\mathcal{O}^{n-1}} (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$.

For $\mathbf{x} \in P^n$, $\mathbf{y} \in P^n_t$, and t, $0 \leqslant t \leqslant r(P^n)$ denote

$$\mathcal{F}^n(\mathbf{x}) = \{\mathbf{z} \in P^n \mid \mathbf{z} \leqslant_{\mathcal{O}^n} \mathbf{x}\},$$

$$\mathcal{F}^n_t(\mathbf{y}) = \{\mathbf{z} \in P^n_t \mid \mathbf{z} \leqslant_{\mathcal{O}^n} \mathbf{y}\} = \mathcal{F}^n(\mathbf{y}) \cap P^n_t,$$

$$\Delta_{\text{new}}(\mathbf{y}) = \Delta(\mathcal{F}^n_t(\mathbf{y})) \backslash \Delta(\mathcal{F}^n_t(\mathbf{y}) \backslash \mathbf{y}).$$

The sets $\mathcal{F}^n(\mathbf{x})$ and $\mathcal{F}^n_t(\mathbf{y})$ are called *initial segments*. We often omit the superscript if $n = 1$. Furthermore, for $n \geqslant 2$ and $A \subseteq P^n$ denote

$$P^n(i,j) = \{(x_1, \ldots, x_n) \in P^n \mid x_i = j\},$$

$$A(i,j) = A \cap P^n(i,j).$$

Obviously, for all $i = 1, \ldots, n$ and any j the subposet of P^n with the element set $P^n(i,j)$ and the induced partial order is isomorphic to P^{n-1}. For $n \geqslant 2$ the set A is called *i-compressed* if for any $j = 0, 1, \ldots, |P| - 1$ the set $A(i,j)$ is an initial segment in order $\mathcal{O}^{n-1}$. If A is i-compressed for any $i = 1, \ldots, n$ then it is called compressed. We call a total order $\mathcal{O}^n$ on P^n an *MRI-order* if any initial segment of this order is a solution to the MRI problem.

Lemma 1. *Let $\mathcal{O}^n$ for $n \geqslant 2$ be a consistent order and let $A \subseteq P^n$ be an ideal. Furthermore, let $\mathcal{O}^{n-1}$ be an MRI-order. Then there exists a compressed ideal $I \subseteq P^n$ such that $|I| = |A|$ and $R(I) \geqslant R(A)$.*

Proof. Let $A \subseteq P^n$ be an ideal. Fix i, $1 \leqslant i \leqslant n$, and consider the set $B \subseteq P^n$ obtained by replacing $A(i,j)$ with the initial segment in order $\mathcal{O}^{n-1}$ of the same size in the corresponding subposet $P^n(i,j)$ for $j = 0, \ldots, p$. Then B is i-compressed.

First we show that B is an ideal. Let $\mathbf{y} = (y_1, \ldots, y_n) \in B$ and $\mathbf{x} = (x_1, \ldots, x_n) <_{P^n} \mathbf{y}$. We have to show $\mathbf{x} \in B$. Without loss of generality, we can assume $r_{P^n}(\mathbf{x}) = r_{P^n}(\mathbf{y}) - 1$. That is, $\mathbf{x}$ and $\mathbf{y}$ differ just in one entry, say the jth one and $r_P(x_j) = r_P(y_j) - 1$. If $j \neq i$ then $\mathbf{x}, \mathbf{y} \in P^n(i, x_i)$ and $\mathbf{x} \in B$ by the consistency of the order $\mathcal{O}^n$. If $j = i$, let us turn back to the set A. Since A is an ideal, for any $\mathbf{z} \in A(i, y_i)$ and $\mathbf{z}'$ obtained from $\mathbf{z}$ by replacing $z_i = y_i$ with $z_i = x_i$ one has $\mathbf{z}' \in A(i, x_i)$. This implies $|A(i, x_i)| \geqslant |A(i, y_i)|$. A similar inequality is satisfied for B. Since $B(i, x_i)$ and $B(i, y_i)$ are initial segments of the same total order $\mathcal{O}^{n-1}$, one has $\mathbf{x} \in B$.

Taking into account (1) one has

$$\begin{aligned} R(A) &= \sum_{\mathbf{x} \in A} \sum_{j=1}^{n} r_P(x_j) = \sum_{\mathbf{x} \in A} \sum_{j \neq i} r_P(x_j) + \sum_{x_i \in P} r_P(x_i) \cdot |A(i, x_i)| \\ &\leqslant \sum_{\mathbf{x} \in B} \sum_{j \neq i} r_P(x_j) + \sum_{x_i \in P} r_P(x_i) \cdot |A(i, x_i)| = R(B). \end{aligned}$$

We apply this operation for $i = 1, 2, \ldots, n$ in the cyclic order until we get an i-compressed set for any i. The resulting set is an ideal and the proof follows. $\square$

For $m = 1, \ldots, p+1$ denote $R(m) = \max_{|I|=m} R(I)$, where the maximum runs over all ideals of P. Let $R(0) = 0$ and for $0 \leqslant m \leqslant p$ denote $\delta_m = R(m+1) - R(m)$.

Lemma 2. *Let $\mathcal{O}$ be an MRI-order on P. Then for $n \geqslant 1$ and for any compressed ideal $I \subseteq P^n$,*

$$R(I) = \sum_{(x_1,\ldots,x_n)\in I} \sum_{i=1}^{n} \delta_{x_i}. \tag{2}$$

Proof. We prove this by induction on $m = |I|$. For $m = 1$ one has $I = \{(0, 0, \ldots, 0)\}$ and the lemma is true. Assume it is true for all ideals of size m and consider an ideal $I \subseteq P^n$ with $|I| = m+1$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in I$ be a maximal element, i.e. such that $\mathbf{y} \notin I$ for any $\mathbf{y}$ larger than $\mathbf{x}$ in the partial order of P^n. Then $I' = I \backslash \{\mathbf{x}\}$ is an ideal and (2) is valid for I' by induction. Note that the rank of $\mathbf{x}$ in P^n is $r(x_1) + \cdots + r(x_n)$. Since $\mathcal{O}$ is an MRI-order for P, one has $r(z) = \delta_z$ for any $z \in P$. This completes the proof. $\square$

If P is a Macaulay poset then the MRI problem always has nested solutions. To specify them we need the following definition. A total order $\mathcal{O}$ on $(P, \leqslant_P)$ is called *rank-greedy* if the following two conditions are satisfied: (i) $\mathcal{O}$ is a linear extension of the partial order $\leqslant_P$ on P and (ii) if $r(x) > r(y)$ and $z <_{\mathcal{O}} y$ for any $z \in \Delta(x)$ then $x <_{\mathcal{O}} y$. It is easily shown that any initial segment of a rank-greedy order is an ideal in P.

For any Macaulay poset there exists a rank-greedy Macaulay order. Indeed, let $(P, \leqslant_P)$ be a Macaulay poset with a Macaulay order $\mathcal{O}$. We define a new total order $\mathcal{O}'$ on P as follows: set the first element of P_0 in order $\mathcal{O}$ to be the first element in order $\mathcal{O}'$; assume $m \geqslant 1$ elements are already ordered in $\mathcal{O}'$ and denote their set by B. Consider

$$A = \{x \in P \backslash B \mid \Delta(x) \subseteq B\}.$$

We now let the next element in order $\mathcal{O}'$ be the smallest (in order $\mathcal{O}$) element of A of maximum rank. Since the restrictions of $\mathcal{O}$ and $\mathcal{O}'$ on P_t is the same order (on P_t) for any t, then the order $\mathcal{O}'$ is Macaulay. It is easily shown that $\mathcal{O}'$ is rank-greedy (see, e.g. [11]) and $(\mathcal{O}')' \equiv \mathcal{O}'$.

Lemma 3 (see Engel [11]). *Let P be a Macaulay poset with a rank-greedy Macaulay order $\mathcal{O}$. Then $\mathcal{O}$ is an MRI-order.*

Note that if $\mathcal{O}$ is an MRI-order for P then P might be Macaulay with some other total order different from $\mathcal{O}$. On the other hand, examples show that P might not be Macaulay (see [3]).

We call a poset P *connected* if its Hasse diagram is a connected graph. For a connected poset $(P, \leqslant_P)$ denote by P^* the *dual* of P, that is, a poset with the same element set and the reversed partial order denoted by $\leqslant_P^*$. It is easily shown that $(P^*)^n \equiv (P^n)^*$.

Lemma 4 (see Engel [11]). *Let P be a connected Macaulay poset with a Macaulay order $\mathcal{O}$. Then P^* is Macaulay and $\mathcal{O}^*$ is a Macaulay order. Moreover, $\mathcal{O}$ is rank-greedy if and only if $\mathcal{O}^*$ is rank-greedy.*

3. The zigzag order as a Macaulay order

Let $(P, \mathscr{Z}^1)$ with $P = \{0, 1, \dots, p\}$ be a Macaulay poset whose elements are numbered according to a Macaulay order $\mathscr{Z}^1$. To simplify matters we define the zigzag order for $|P| \geqslant 4$ only. It is not a significant restriction, because any cartesian power of P is Macaulay if $|P| \leqslant 3$. Such posets are represented by the Boolean lattice, the star poset and its dual, and lattice of multisets of size 3.

For $n \geqslant 2$ we define the *zigzag* order $\mathscr{Z}^n$ on P^n (with $p \geqslant 3$) by specifying the successor of each element of P^n as follows:

1. $\text{succ}(0, b_2, \dots, b_n) = (1, b_2, \dots, b_n)$.
2. $\text{succ}(1, b_2, \dots, b_n) = (0, \text{succ}(b_2, \dots, b_n))$ if $(b_2, \dots, b_n) \neq (p, \dots, p)$, and $\text{succ}(1, p, \dots, p) = (2, 0, \dots, 0)$.
3. If $2 \leqslant b_1 < p - 1$ then $\text{succ}(b_1, b_2, \dots, b_n) = (b_1, \text{succ}(b_2, \dots, b_n))$ if $(b_2, \dots, b_n) \neq (p, \dots, p)$, and $\text{succ}(b_1, p, \dots, p) = (b_1 + 1, 0, \dots, 0)$.
4. $\text{succ}(p - 1, b_2, \dots, b_n) = (p, b_2, \dots, b_n)$.
5. $\text{succ}(p, b_2, \dots, b_n) = (p - 1, \text{succ}(b_2, \dots, b_n))$ if $(b_2, \dots, b_n) \neq (p, \dots, p)$, otherwise there is no successor.

Therefore, the minimum and the maximum elements of P^n are $(0, \dots, 0)$ and $(p, \dots, p)$, respectively. Since any element different from $(p, \dots, p)$ has a uniquely defined successor, the total order on P^n is well-defined. This order can also be defined in an equivalent way that we will use in our analysis. It follows from above that $(a_1, \dots, a_n) >_{\mathscr{Z}^n} (b_1, \dots, b_n)$ if and only if

1. $a_1 - b_1 \geqslant 2$, or
2. $a_1 - b_1 = 1$ and $b_1 \notin \{0, p - 1\}$, or
3. $a_1 - b_1 = 1$ and $b_1 \in \{0, p - 1\}$ and $(a_2, \dots, a_n) \geqslant_{\mathscr{Z}^{n-1}} (b_2, \dots, b_n)$, or
4. $a_1 = b_1$ and $(a_2, \dots, a_n) >_{\mathscr{Z}^{n-1}} (b_2, \dots, b_n)$, or
5. $a_1 - b_1 = -1$ and $a_1 \in \{0, p - 1\}$ and $(a_2, \dots, a_n) >_{\mathscr{Z}^{n-1}} (b_2, \dots, b_n)$.

It is easy to show that the dual of $\mathscr{Z}^n$ is isomorphic to $\mathscr{Z}^n$. Thus, by Lemma 4 if $\mathscr{Z}^n$ is Macaulay for P^n then $\mathscr{Z}^n$ is Macaulay for $(P^*)^n$. We will often use this assertion in our analysis.

Lemma 5 (see Carlson [9]). *For any $n \geqslant 2$ the order $\mathscr{Z}^n$ is consistent.*

Although this lemma is proved in [9] for $p = 4$ only, the proof is nearly identical for $p > 4$.

A poset $(P, \leqslant_P)$ is called *graded* if every minimal element has rank 0 and every maximal element has the same rank. The proof of the next technical lemma is moved to Section 5, because it is based on the concept of additivity that first appears in that section.

Lemma 6. *If the order $\mathscr{L}^2$ is Macaulay for P^2 and P is connected, then P is graded.*

The next lemma will be used in the proof of Lemma 8.

Lemma 7. *Let $\mathscr{L}^2$ and $\mathscr{L}^3$ be Macaulay orders on P^2 and P^3, respectively. Furthermore, let $|P| \geqslant 4$ and P be connected. Then either $|P_0| = |P_{r(P)}| = 1$ or P is the N-poset (see Fig. 4(b)).*

Proof. Let $t = r(P)$ and assume $|P_t| > 1$. Denote by b the minimum element of P_t and let $c \in P_t$ with $c >_{\mathscr{L}^1} b$. Let $a \in \Delta(b)$.

Case 1: Assume $a <_{\mathscr{L}^1} b$. Since (b, b) is the first element of P^2_{2t} in order $\mathscr{L}^2$ then $\Delta((b, b))$ must be an initial segment. If $a \neq 0$ or $b \neq 1$ one has $(a, c) <_{\mathscr{L}^2} (b, a)$. Since $(b, a) \in \Delta((b, b))$ and $(a, c) \notin \Delta((b, b))$, we get a contradiction with the continuity property. Thus, we can assume $a = 0$ and $b = 1$.

Fact 1. *If $0 \in \Delta(1)$ then $r(0) = 0$.*

Proof. Assume $r(0) > 0$ and let $x \in \Delta(0)$ for some $x \geqslant 2$. Such an element x does exist due to Lemma 6. Since the first four elements of P^3 in the order $\mathscr{L}^3$ are the elements $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, for the set $A = \mathscr{F}^3_{3t-1}((1,1,0))$ one has $|A| = 1$. Furthermore, $(1,1,x) \in \Delta(A)$, $(0,0,1) <_{\mathscr{L}^3} (1,1,x)$ and $(0,0,1) \notin \Delta(A)$, contradicting the continuity property. $\square$

Since $0 \in \Delta(1)$ and $r(1) = r(P)$, Fact 1 implies $r(P) = 1$. Now, we show

$$\Delta(1) = \{0\}. \tag{3}$$

To show this assume $x \in \Delta(1)$ for some $x \geqslant 2$ then consider $A = \mathscr{F}^2_1((1,0))$. One has $|A| = 1$, $(x, 0) \in \Delta(A)$, $(0, 1) <_{\mathscr{L}^2} (x, 0)$ and $(0, 1) \notin \Delta(A)$, contradicting the continuity property.

Case 1a: Assume $|P_0| > 1$ and let $z \in P_0$ be the minimum element with $z >_{\mathscr{L}^1} 0$. Let $d \in P_1$ be the first element such that $z \in \Delta(d)$. Since P is connected, the element d does exist. We show that d is the first element such that $d >_{\mathscr{L}^1} 1$ and $d \in P_1$. For this assume the contrary and let $C = \{x \in P_1 \mid 1 <_{\mathscr{L}^1} x <_{\mathscr{L}^1} d\}$. Let $k = |C| \geqslant 1$ and let $x_1 <_{\mathscr{L}^1} x_2 <_{\mathscr{L}^1} \cdots <_{\mathscr{L}^1} x_k$ be the elements of C. Consider the set $A = \{(1,1), (1, x_1), \ldots, (1, x_k), (1, d)\} \subset P^2_2$, which is an initial segment. By (3), $\{(0, d), (1, z)\} \subseteq \Delta_{\text{new}}((1, d))$ and $\Delta((x_1, 1)) \backslash \Delta(A \backslash \{(1, d)\}) = \{(x_1, 0)\}$. Hence, for

$B = (A \backslash \{(1,d)\}) \cup \{(x_1,1)\}$ one has $|\Delta(B)| \leqslant |\Delta(A)| - 1$, contradicting the optimality of A. This implies, $C = \emptyset$.

Now, consider the set $D = \mathscr{F}_2^2(d,1)$. If $d >_{\mathscr{L}^1} z$ and $d \neq p$ one has $(d,0) \in \Delta(D)$, $(z,d) <_{\mathscr{L}^2} (d,0)$ and $(z,d) \notin \Delta(D)$, which contradicts the continuity. Therefore, $d = p$ or $d <_{\mathscr{L}^1} z$.

If $d = p$ then $|P_1| = 2$. Note that the assumptions of case 1a are satisfied for the dual poset P^*. Applying the above arguments to P^*, one gets $|P_0| = |P_1^*| = 2$. This and (3) imply P is the N-poset.

If $d <_{\mathscr{L}^1} z$ and $z \neq p$ one has $(z,1) \in \Delta(D)$, $(d,z) <_{\mathscr{L}^2} (z,1)$ and $(d,z) \notin \Delta(D)$, which contradicts the continuity. If $z = p$ then $|P_0| = |P_1| = 2$ and using (3) again we get that P is the N-poset.

Case 1b: Assume $|P_0| = 1$. Since $|P| \geqslant 4$ we have $|P_1| \geqslant 3$ and $0 <_{\mathscr{L}^1} i$ for $i = 1,2,3$. Now, let E be the initial segment of length 4 in P_2^2. One has $E = \{(1,1),(1,2),(1,3),(1,4)\}$ for $|P_1| > 3$ and $E = \{(1,1),(1,2),(1,3),(2,1)\}$ for $|P_1| = 3$. In both cases $|\Delta(E)| = 5$. However, for the set $F = \{(1,1),(1,2),(2,1),(2,2)\}$ one has $|\Delta(F)| = 4$. Since F is not an initial segment we have a contradiction with the nestedness property.

Case 2: Assume $a >_{\mathscr{L}^1} b$. Similar to the above, (b,b,b) is the first element of P_{3t}^2, so its shadow has to be an initial segment. However, if $a \neq 1$ or $b \neq 0$ one has $(b,c,a) \notin \Delta(b,b,b)$ and $(b,c,a) <_{\mathscr{L}^3} (a,b,b) \in \Delta(b,b,b)$, contradicting the continuity. From now on assume $a = 1$ and $b = 0$.

Fact 2. *If* $1 \in \Delta(0)$ *then* $r(1) = 0$.

Proof. Assume $r(1) > 0$ and let $x \in \Delta(1)$ for some $x \geqslant 2$. Such an element x does exist due to Lemma 6. Then for $A = \mathscr{F}_{2t-1}^2((1,0))$ one has $|A| = 1$, $(x,0) \in \Delta(A)$, $(0,x) <_{\mathscr{L}^2} (x,0)$ and $(0,x) \notin \Delta(A)$, contradicting the continuity. □

Since $1 \in \Delta(0)$ and $r(0) = r(P)$, Fact 2 implies $r(P) = 1$. Similar to the above we can show $\Delta(0) = \{1\}$. Namely, if $x \in \Delta(0)$ and $x \geqslant_{\mathscr{L}^1} 2$ then consider the set $A = \mathscr{F}_2^2((0,0))$. One has $|A| = 1$, $(x,0) \in \Delta(A)$, $(1,c) <_{\mathscr{L}^2} (x,0)$ and $(1,c) \notin \Delta(A)$, contradicting the continuity property. Now the rest of the proof goes along the lines of cases 1a and 1b by exchanging 0 and 1 in all arguments. □

This lemma is obviously true for $|P| = 2$ but not for $|P| = 3$. In the later case the first assumption of the lemma is satisfied for the star poset with one element in P_0 and two elements in P_1. However, this poset and its dual are among few exceptions.

A lemma similar to the following lemma has already appeared in [7] with respect to the lexicographic order. The proof below is essentially borrowed from [7] and modified mostly for the cases when the lexicographic order and the zigzag order do not match. Lemmas 8 and 9, the key lemmas for our technique, and Lemma 3 allow us to approach the SMP through the MRI problem.

Lemma 8. *Assume P is a connected poset different from the N-poset and let $|P| \geqslant 4$. Furthermore, let $\mathscr{L}^2$, $\mathscr{L}^3$, and $\mathscr{L}^4$ be Macaulay orders on P^2, P^3, and P^4, respectively. Then the order $\mathscr{L}^1$ on P is rank-greedy.*

Proof. First assume the order $\mathscr{L}^1$ is not a linear extension of $\leqslant_P$, i.e. there exist $a, b \in P$ such that $a \in \Delta(b)$ and $a >_{\mathscr{L}^1} b$. We call such pair $\{a, b\}$ an *inverted pair* and show that the existence of an inverted pair implies that P is a chain. The proof is long, and is preceded by four facts.

Denote by f_i (resp. l_i) the first (resp. last) element of P_i, $i = 0, \ldots, r(P)$.

Fact 3. *For any $z \in P_i$, $i = 1, \ldots, r(P)$, $\Delta(z) = P_{i-1}$.*

Proof. Let us choose an inverted pair $\{a, b\}$ satisfying

$$\text{if } b \in \Delta(x) \text{ then } b <_{\mathscr{L}^1} x. \tag{4}$$

Since P is finite and (4) is trivially satisfied if $r(b) = r(P)$, it is easily shown that an inverted pair satisfying (4) does exist. Assume to the contrary that there exists an element $u \in P_{i-1} \backslash \Delta(f_i)$ for some $i \geqslant 1$. Let $t = r_P(b)$ and denote $B = \mathscr{F}^2_{t+i}(b, f_i) \subseteq P^2_{t+i}$. Then $(b, u) \notin \Delta(B)$. Indeed, if $(b, u) \in \Delta(x, y)$ for some $(x, y) \in B$ then either $b \in \Delta(x)$ or $u \in \Delta(y)$.

If $b \in \Delta(x)$ then $b <_{\mathscr{L}^1} x$ by (4). Since $t \geqslant 1$, Fact 1 applied to the pair $b = 0$ and x shows that $x \neq 1$. This implies $(b, f_i) <_{\mathscr{L}^2} (x, y) = (x, f_i)$. Furthermore, if $u \in \Delta(y)$ then $f_i <_{\mathscr{L}^1} y$ and $x = b$. This implies $(b, f_i) <_{\mathscr{L}^2} (x, y) = (b, y)$ again, and $(b, u) \notin \Delta(B)$ is established.

Now, if $(a, b) \notin \{(1, 0),\ (p, p-1)\}$ then $(b, u) <_{\mathscr{L}^2} (a, f_i) \in \Delta(B)$. Since B is an initial segment, we have a contradiction with the continuity property. Consider the case $(a, b) = (1, 0)$. Fact 2 implies $r(1) = 0$ and $r(0) = 1$. Let $A = \{(1,1,0,0),\ (1,0,1,0),\ (0,1,1,0)\} \subseteq P^4_2$. Then A is an initial segment and $|\Delta(A)| = 4$. However, for the set $B = \{(1,1,0,0),\ (1,0,1,0),\ (1,0,0,1)\}$ one has $|\Delta(B)| = 3$, contradicting the Macaulayness of P^4.

If $(a, b) = (p, p-1)$ then we have an inverted pair $(1, 0)$ in the Macaulay poset P^*. This leads to a contradiction with the Macaulayness of $(P^*)^4$ as it is shown in the last paragraph. Therefore, the element u does not exist, i.e. $\Delta(f_i) = P_{i-1}$ for $i = 1, \ldots, r(P)$. This and the Macaulayness of P imply the assertion for any $z \in P_i$. $\square$

Fact 4. *If $r(0) = r(1)$ then $\Delta(0) = \emptyset$.*

Proof. Assume $\Delta(0) \neq \emptyset$, which by Lemma 6 is equivalent to $s = r(0) > 0$. Let $x \in \Delta(0)$ for some $x \geqslant_{\mathscr{L}^1} 2$. Note that $(0, 0)$ is the first element in its level of P^2. Consider $A = \mathscr{F}^2_{2s}((0, 0))$. One has $|A| = 1$, $(x, 0) \in \Delta(A)$, $(1, x) <_{\mathscr{L}^2} (x, 0)$, but $(1, x) \notin \Delta(A)$, contradicting the continuity property. $\square$

Fact 5. *If $\{a, b\}$ is an inverted pair and $t = r(b)$, then $a = l_{t-1}$ and $b = f_t$.*

Proof. First we show that $a = l_{t-1}$. Since this is true if $|P_{t-1}| = 1$, let us assume $|P_{t-1}| > 1$. Since $f_t \leqslant_{\mathscr{L}^1} b$, (a, f_t) is also an inverted pair (Fact 3). So, we can assume $b = f_t$. Suppose $a <_{\mathscr{L}^1} l_{t-1}$ and consider $A = \mathscr{F}^2_{2t-1}((b, a))$.

We show $(a, l_{t-1}) \notin \Delta(A)$. Indeed, assume $(a, l_{t-1}) \in \Delta((x, y))$ for some $(x, y) \in A$. Then either $x = a$ or $y = l_{t-1}$. If $y = l_{t-1}$ then $r(x) = t$, so $x \geqslant_{\mathscr{L}^1} b$ by the choice of b. Since $(a, l_{t-1}) \notin \Delta((b, a))$ then $x >_{\mathscr{L}^1} b$. Now, if $(b, x) \neq (0, 1)$ and $(b, x) \neq (p-1, p)$ then $(x, l_{t-1}) >_{\mathscr{L}^2} (b, a)$, so $(x, y) \notin A$, a contradiction. If $(b, x) = (0, 1)$ then, by Fact 2, this contradicts the existence of a. Since $b <_{\mathscr{L}^1} a$, $(b, x) = (p-1, p)$ is impossible.

Consider the case $x = a$. Now, $l_{t-1} >_{\mathscr{L}^1} a$ implies $(a, b) \neq (p, p-1)$. Furthermore, if $(a, b) \neq (1, 0)$ then $(x, y) = (a, y) >_{\mathscr{L}^2} (b, a)$, so $(x, y) \notin A$. Finally, if $(a, b) = (1, 0)$ then $r(a) = 0$ by Fact 5. Lemma 7 implies $|P_0| = 1$ which contradicts $a \neq l_{t-1}$.

Now we show $(a, l_{t-1}) <_{\mathscr{L}^2} (l_{t-1}, a)$. This is obviously true if $(a, l_{t-1}) \neq (0, 1)$ and $(a, l_{t-1}) \neq (p, p-1)$. Since $b <_{\mathscr{L}^1} a$ then $a \geqslant 1$. Consider the case $(a, l_{t-1}) = (p, p-1)$. Now in the dual poset P^* the elements 0 and 1 are in the same level. This leads to a contradiction provided by Fact 4.

Therefore, $(a, l_{t-1}) \notin \Delta(A)$ and $(a, l_{t-1}) <_{\mathscr{L}^2} (l_{t-1}, a) \in \Delta(A)$, thus contradicting the continuity property. Hence, $a = l_{t-1}$. The second part of the statement follows by applying similar arguments to the dual poset $(P^*)^2$. □

Fact 6. *If $\{a, b\}$ be an inverted pair and $t = r(b)$, then $|P_t| = |P_{t-1}| = 1$.*

Proof. Suppose $|P_t| > 1$. By Fact 4, $(b, l_t) \neq (0, 1)$. Since $a >_{\mathscr{L}^1} b$ and $l_t >_{\mathscr{L}^1} b$ then $b \neq p - 1$. Furthermore, by Lemma 7, $t < r(P)$. Fact 3 implies $b = f_t \in \Delta(x)$ for any $x \in P_{t+1}$, and it follows from Fact 5 that $b <_{\mathscr{L}^1} x$. To get a contradiction consider $A = \mathscr{F}^3_{3t}((b, b, b))$.

First we show that $(b, l_t, a) \notin \Delta(A)$. Indeed, assume that $(b, l_t, a) \in \Delta(x, y, z)$ for some $(x, y, z) \in A$. Then the vectors (b, l_t, a) and (x, y, z) differ in only one entry. If $b \in \Delta(x)$ then $x >_{\mathscr{L}^1} b$ by above. So if $(b, x) \neq (0, 1)$ then $(x, l_t, a) >_{\mathscr{L}^3} (b, b, b)$. If $(b, x) = (0, 1)$ then $r(b) = 0$ by Fact 1, which contradicts the existence of a. If $l_t \in \Delta(y)$ then $y >_{\mathscr{L}^1} b$ and similar argument as above implies $(b, y, a) >_{\mathscr{L}^3} (b, b, b)$. Finally, if $a \in \Delta(z)$ then $(b, l_t) \neq (0, 1)$ (Fact 4) and $l_t >_{\mathscr{L}^1} b$ imply $(b, l_t, z) >_{\mathscr{L}^3} (b, b, b)$, a contradiction. Hence, $(b, l_t, a) \notin \Delta(A)$ is established.

Note again that $|P_t| > 1$ implies $(b, a) \neq (p-1, p)$. Now, if $(b, a) \neq (0, 1)$ then $(b, l_t, a) <_{\mathscr{L}^3} (a, b, b) \in \Delta(A)$, contradicting the continuity property. If $(b, a) = (0, 1)$ then we come to a contradiction with the Macaulayness of P^4 as it is shown in the proof of Fact 3. Similar arguments in the dual posets $(P^*)^2$ and $(P^*)^3$ imply $|P_{t-1}| = 1$. □

We are now ready to conclude the proof of the Lemma. Let (a, b) be an inverted pair and let $t = r(b)$. Without loss of generality we can assume $r(P) \geqslant 2$, since otherwise the inverted pair $(1, 0)$ contradicts the Macaulayness of P^4. Assume additionally $t < r(P)$ and $f_{t+1} >_{\mathscr{L}^1} b$. Consider $A = \mathscr{F}_{t+1}((b, f_1)) \subseteq P^2_{t+1}$. Now, $B = \Delta_{\text{new}}((b, f_1)) = \{(b, f_0), (a, f_1)\}$. Furthermore, $(f_{t+1}, f_0) \notin A$ and $\Delta(f_{t+1}, f_0) = \{(b, f_0)\}$.

Therefore,

$$|\Delta((A\backslash(b,f_1))\cup(f_{t+1},f_0))| = |\Delta(A)| - |B| + 1 < |\Delta(A)|,$$

which contradicts the optimality of A.

Hence, $f_{t+1} <_{\mathscr{L}^1} b$ and, thus, $\{b,f_{t+1}\}$ is an inverted pair. Fact 6 implies $|P_{t+1}| = 1$. By iterating the argument with respect to the inverted pair $\{f_{t+i},f_{t+i+1}\}$ for $1 \leqslant i \leqslant r(P) - t - 1$ we get

$$b = f_t >_{\mathscr{L}^1} f_{t+1} >_{\mathscr{L}^1} \cdots >_{\mathscr{L}^1} f_{r(P)} \quad \text{and} \quad |P_t| = |P_{t+1}| = \cdots |P_{r(P)}| = 1.$$

Similar arguments in the dual $(P^*)^2$ provide

$$f_0 >_{\mathscr{L}^1} f_1 >_{\mathscr{L}^1} \cdots >_{\mathscr{L}^1} f_{t-1} = a \quad \text{and} \quad |P_0| = |P_1| = \cdots |P_{t-1}| = 1.$$

Therefore, P is a chain and $f_0 >_{\mathscr{L}^1} \cdots >_{\mathscr{L}^1} f_p$ for $p \geqslant 2$. If $p \geqslant 3$ then $P_2^2 = \{(p-2,p),(p,p-2),(p-1,p-1)\}$, where the elements are listed in increasing order $\mathscr{L}^2$. Consider $B = \mathscr{F}_2^2((p-2,p))$. One has $|B| = 1$ and $\Delta(B) = \{(p-1,p)\}$ is not an initial segment, since $(p,p-1) \notin \Delta(B)$. This contradicts the continuity property. For $p = 2$ one has $P_2^2 = \{(1,1),(0,2),(2,0)\}$. Denoting $A = \{(1,1)\}$ and $B = \{(0,2)\}$ we have $|\Delta(A)| = 2 > 1 = |\Delta(B)|$, which is a contradiction.

The above arguments show that the Macaulay order $\mathscr{L}^1$ is a linear extension of the partial order $\leqslant$. To complete the proof we have to show that the second claim in the definition of rank-greediness is fulfilled for P. Suppose this is not true. Then there exist $a,b \in P$ such that $z <_{\mathscr{L}^1} a$ for any $z \in \Delta(b)$, $r(b) > r(a)$ and $b >_{\mathscr{L}^1} a$. Since $\Delta(b) \neq \emptyset$ then $a \neq 0$. So, $(b,a) \neq (1,0)$. Furthermore, $b \neq p$ since otherwise b is the top element of P because $\mathscr{L}^1$ is a linear extension of $\leqslant$. Hence, $(b,a) \neq (p,p-1)$.

Let $q = r(b) - r(a)$ and consider the set $B = \mathscr{F}^2_{r(a)+q}((a,f_q)) \subseteq P^2_{r(a)+q}$. Since $(b,f_0) >_{\mathscr{L}^2} (a,f_q)$, then $(b,f_0) \notin B$. Furthermore, $z <_{\mathscr{L}^1} a$ for any $z \in \Delta(b)$ implies $\Delta((b,f_0)) \subseteq \Delta(B)$. Since $a \neq b$ and $q \geqslant 1$, then $\Delta((b,f_0)) \cap \Delta((a,f_q)) = \emptyset$.

Since the order $\mathscr{L}^1$ is a linear extension of $\leqslant$, and since $q \geqslant 1$, then $(a,f_{q-1}) \in \Delta_{\text{new}}((a,f_q))$. Thus, $|\Delta_{\text{new}}((a,f_q))| \geqslant 1$. One has

$$|\Delta((B\backslash(a,f_q))\cup(b,f_0))| = |\Delta(B)| - |\Delta_{\text{new}}((a,f_q))| < |\Delta(B)|,$$

which contradicts the optimality of B. □

It is worth noting that Lemma 8 is not necessarily true without the assumptions concerning $n = 3$ and 4. The diamond poset P shown in Fig. 1(a) has the Macaulay order presented in the figure which is not rank-greedy, and $\mathscr{L}^2$ is Macaulay for P^2. Furthermore, $\mathscr{L}^2$ and $\mathscr{L}^3$ are Macaulay for the corresponding powers of the N-poset and the 2-cube with the Macaulay order shown in Fig. 1(b) and (c), respectively, but Z^1 is not rank-greedy.

Lemma 9. *If* $|P_0| = |P_{r(P)}| = 1$, *the order* $\mathscr{L}^1$ *is rank-greedy on P and* $\mathscr{L}^2$ *is Macaulay for* P^2, *then* $\mathscr{L}^n$ *is rank-greedy on* P^n *for any* $n \geqslant 2$.

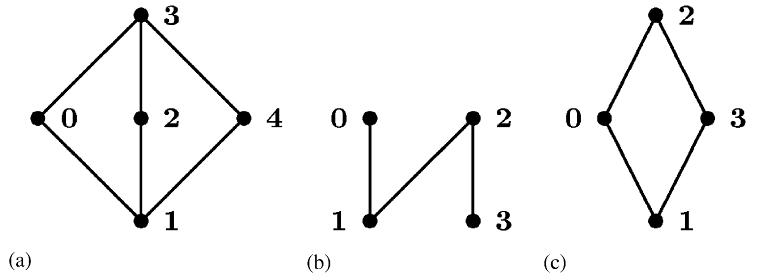

Fig. 1. Counterexamples to Lemma 8. (a) $n=2$. (b and c) $n=3$.

Proof. To show that $\mathscr{L}^n$ is a linear extension of P^n consider $\mathbf{x},\mathbf{y}\in P^n$ with $\mathbf{x}<_{P^n}\mathbf{y}$. We have to show $\mathbf{x}<_{\mathscr{L}^n}\mathbf{y}$. Without loss of generality, we can assume $r_{P^n}(\mathbf{x})=r_{P^n}(\mathbf{y})-1$. Hence, $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{y}=(y_1,\dots,y_n)$ differ in one entry, say the ith one. Therefore, $x_i\in\Delta(y_i)$ in P. Since $\mathscr{L}^n$ is consistent then $\mathbf{x}<_{\mathscr{L}^n}\mathbf{y}$ if and only if $x_i<_{\mathscr{L}^1}y_i$. But the last inequality holds due to the rank-greediness of $\mathscr{L}^1$.

To prove the second claim in the definition of rank-greediness we apply the induction on n. For $n=1$ this is true, so we proceed with $n\geqslant 2$. Suppose that there exist $\mathbf{x},\mathbf{y}\in P^n$ with $r_{P^n}(\mathbf{x})>r_{P^n}(\mathbf{y})$ and $\mathbf{x}>_{\mathscr{L}^n}\mathbf{y}$ such that

$$\mathbf{z}<_{\mathscr{L}^n}\mathbf{y}\quad\text{for any } \mathbf{z}\in\Delta(\mathbf{x}). \tag{5}$$

Case 1: Assume there exists a $z\in P$ such that $x_1>_{\mathscr{L}^1}z>_{\mathscr{L}^1}y_1$. Now, $r(x_i)=0$ for $2\leqslant i\leqslant n$, since otherwise for $z_i\in\Delta(x_i)$ one has $(x_1,\dots,x_{i-1},z_i,x_{i+1},\dots,x_n)>_{\mathscr{L}^n}\mathbf{y}$, contradicting (5). Since $|P_0|=1$, $x_2=\cdots=x_n=0$. Since $r_{P^n}(\mathbf{x})>r_{P^n}(\mathbf{y})$ then $r_P(x_1)>r_P(y_1)$. Since $x_1>_{\mathscr{L}^1}y_1$ and the order $\mathscr{L}^1$ is rank-greedy, there exists $z\in\Delta(x_1)$ such that $z\geqslant_{\mathscr{L}^1}y_1$. Let z be the maximum element satisfying these conditions. Since $z\in\Delta(x_1)$ then $z\neq p$.

Assume $y_1>_{\mathscr{L}^1}0$. Now, if $z>_{\mathscr{L}^1}y_1$ then $(z,x_2,\dots,x_n)>_{\mathscr{L}^n}\mathbf{y}$, which contradicts (5). If $z=y_1$ then $r_{P^n}(\mathbf{x})=r_P(x_1)=r_P(y_1)+1$ and $r_{P^n}(\mathbf{x})>r_{P^n}(\mathbf{y})$ imply $r_{P^n}(\mathbf{x})=r_{P^n}(\mathbf{y})+1$ and $r(y_2)=\cdots=r(y_n)=0$, so $(y_2,\dots,y_n)=(0,\dots,0)$. Hence, $\mathbf{y}\in\Delta(\mathbf{x})$ which contradicts (5).

Assume $y_1=0$. If $z>_{\mathscr{L}^1}1$ then we have a similar contradiction as above. Now, if $z=1$ then $\Delta(x_1)=\{z\}$. If $r_{P^n}(\mathbf{y})=0$ then $\Delta(\mathbf{x})=\{(1,0,\dots,0)\}$ contradicts (5). Therefore, $r_{P^n}(\mathbf{y})=1$ which implies $r_{P^{n-1}}((y_2,\dots,y_n))=1$. Now, if $n\geqslant 3$ then $\mathbf{x}$ and $\mathbf{y}$ have a common zero entry and the lemma is true by induction and the consistency of $\mathscr{L}^n$. If $n=2$ then the first element of P_2^2 is $\mathbf{z}=(1,1)$ and $|\Delta(\mathbf{z})|=2>1=|\Delta(\mathbf{x})|$ contradicting the Macaulayness of $\mathscr{L}^2$.

Case 2: Assume $x_1=y_1+1$, $x_1\neq p$, and $y_1\neq 0$. By similar arguments to the above one has $r(x_i)=0$ for $2\leqslant i\leqslant n$, $r(x_1)>r(y_1)$, and $z>_{\mathscr{L}^1}y_1$ for some $z\in\Delta(x_1)$. Therefore, $y_1<_{\mathscr{L}^1}z<_{\mathscr{L}^1}x_1$. But since x_1 and y_1 are consecutive elements in order $\mathscr{L}^1$ the element z does not exist. The obtained contradiction implies that this case is impossible.

Case 3: Assume $(x_1, y_1) = (1, 0)$ or $(x_1, y_1) = (p, p-1)$. Now $\mathbf{x} >_{\mathscr{Z}^n} \mathbf{y}$ implies $(x_2, \dots, x_n) \geqslant_{\mathscr{Z}^{n-1}} (y_2, \dots, y_n)$. Since P has one minimum and one maximum element, $y_1 \in \Delta(x_1)$. This implies $\mathbf{z} = (0, x_2, \dots, x_n) \in \Delta(\mathbf{x})$ and $\mathbf{z} \geqslant_{\mathscr{Z}^n} \mathbf{y}$ which contradicts (5).

Case 4: Assume $(x_1, y_1) = (0, 1)$ or $(x_1, y_1) = (p-1, p)$. Then $(x_2, \dots, x_n) >_{\mathscr{Z}^{n-1}} (y_2, \dots, y_n)$ and $r_{P^{n-1}}(x_2, \dots, x_n) > r_{P^{n-1}}(y_2, \dots, y_n)$. By induction there exists an element $(z_2, \dots, z_n) \in \Delta((x_2, \dots, x_n))$ such that $(z_2, \dots, z_n) \geqslant_{\mathscr{Z}^{n-1}} (y_2, \dots, y_n)$. But then for $\mathbf{z} = (x_1, z_2, \dots, z_n) \in \Delta(\mathbf{x})$ one has $\mathbf{z} \geqslant_{\mathscr{Z}^n} \mathbf{y}$ contradicting (5). □

Later in this section we will obtain necessary conditions for a poset P in order for the zigzag order $\mathscr{Z}^n$ to be Macaulay for P^n for any $n \geqslant 2$. If $\mathscr{Z}^n$ is Macaulay for P^n then by Lemma 3 it is an MRI-order. The MRI problem is easier to analyze than the SMP. As we show there exists a limited number of posets that satisfy this necessary condition. The proof that these posets are Macaulay will be done in Sections 4 and 5. Note that the proof of Lemma 10 and Theorem 1 can be slightly simplified by adding an additional condition concerning $\mathscr{Z}^3$ and using Lemma 7. However, we did not do this in order to present a method that can be applied to a more general situation.

Throughout this section we assume the MRI problem on a connected poset P has nested solutions provided by some total order $\mathscr{Z}^1$ and that order $\mathscr{Z}^2$ is an MRI-order for P^2. For any integer $n \geqslant 1$ and $m \geqslant 0$ denote by $\mathscr{F}^n(m) \subseteq P^n$ the initial segment of length m of the order $\mathscr{Z}^n$ on P^n. Let $\delta(P) = (\delta_0, \dots, \delta_p)$.

Lemma 10. *Let P with $|P| \geqslant 4$ be a connected poset and $\mathscr{Z}^1$ and $\mathscr{Z}^2$ be MRI-orders for P and P^2, respectively. Then the following three conditions hold for $\delta(P)$:*

1. $\delta_{i+1} \leqslant \delta_i + 1$ *for* $0 \leqslant i < p$;
2. $\delta_1 = 1$ *and* $0 \leqslant \delta_2 \leqslant 1$;
3. *If* $\delta_2 = 0$ *then P is the N-poset.*

Proof. To show the first assertion assume the contrary, i.e. $\delta_{i+1} \geqslant \delta_i + 2$ for some $i \geqslant 0$. Let $I_1 = \mathscr{F}^1(i)$, $I_2 = \mathscr{F}^1(i+1)$ and $I_3 = \mathscr{F}^1(i+2)$. One has $I_1 \subset I_2 \subset I_3$. Let $x = I_2 \backslash I_1$ and $y = I_3 \backslash I_2$. Since $\delta_{i+1} - \delta_i \geqslant 2$ then $r(y) - r(x) \geqslant 2$. This implies that either x and y are incomparable, or there exists $z \in P$ with $x <_P z <_P y$. In the later case $z \in I_1$ since x and y are consecutive elements in order $\mathscr{Z}^1$. Therefore, $I = I_1 \cup \{y\}$ is an ideal and $|I| = |I_2| = i + 1$. One has $R(I) = R(I_2) - r(x) + r(y) \geqslant R(I_2) + 2$. This contradicts the fact that $\mathscr{Z}^1$ is an MRI-order.

For the second assertion $\delta_0 = 0$ implies $\delta_1 \leqslant 1$. Assume $\delta_1 = 0$. If $\delta_i = 0$ for $i = 0, \dots, p$ then P is an antichain and, hence is not connected. Otherwise, let $i = \min\{j \mid \delta_j > 0\}$ and consider $A = \{(x, 0) \mid x = 0, 1, \dots, i\} \subseteq P^2$. Then A is an ideal and $R(A) = 1$ by Lemma 2. It is easily seen that $R(\mathscr{F}^2(|A|)) = 0$, a contradiction. Similarly, part (a) implies $\delta_2 \leqslant 2$. Assume $\delta_2 = 2$ and consider the ideal $A = \{(0,0), (0,1), (0,2)\}$. One has $R(A) - R(\mathscr{F}^2(|A|)) = 1$, a contradiction.

Finally, for the last assertion, $i = \min\{j \geqslant 3 \mid \delta_j > 0\}$. If $|P| > 4$ consider the ideal

$$A = (\{0, 1, \dots, i-2\} \times P) \cup \{(i-1, 0), (i-1, 1), (i, 0), (i, 1)\}.$$

One has $R(A) > R(\mathscr{F}^2(|A|))$, which is a contradiction, and so $|P| \geqslant 4$ implies that $|P| = 4$. By part (a) one has $0 \leqslant \delta_3 \leqslant 1$. Therefore, either $\delta(P) = (0, 1, 0, 1)$ or $\delta(P) = (0, 1, 0, 0)$. In the first case P is the N-poset, while in the second case P is not connected. $\square$

Denote by H^n the Boolean lattice (or the Hypercube) of dimension n. We need the last auxiliary lemma before proving Theorem 1.

Lemma 11. *If $\mathscr{L}^2$ is an MRI-order on P^2 then the lexicographic order is an MRI-order on $P \times H^k$ for any $k \geqslant 1$.*

Proof. The lemma is true for $k = 1$ since $P \times H^1$ with the lexicographic order is isomorphic to an initial segment of $\mathscr{L}^2$ on P^2. Assume $k \geqslant 2$, and let A be a compressed set which is a solution to the MRI problem. Let $\mathbf{a} = (a, \alpha_1, \dots, \alpha_k)$ be the last element (in lexicographic order) of A and let $\mathbf{b} = (b, \beta_1, \dots, \beta_k)$ be the first element of $(P \times H^k) \backslash A$. Without loss of generality we can assume that $\mathbf{a}$ and $\mathbf{b}$ have no equal entry. If $a <_{\mathscr{L}^1} b$ then A is an initial segment of the lexicographic order, so let us assume $b <_{\mathscr{L}^1} a$.

If $\alpha_i = 1$ for some i, $1 \leqslant i \leqslant k$, then consider $\mathbf{c}$ obtained from $\mathbf{a}$ by replacing $\alpha_i = 1$ with $\alpha_i = 0$. Observe that $\mathbf{b} <_{\text{lex}} \mathbf{c} <_{\text{lex}} \mathbf{a}$. Now, $\mathbf{a}$ and $\mathbf{c}$ have an equal entry, so $\mathbf{c} \in A$. By a similar reason $\mathbf{b} \in A$, a contradiction. So, without loss of generality we can assume that $\alpha_i = 0$ and $\beta_i = 1$ for all $1 \leqslant i \leqslant k$.

If there exists c such that $b <_{\mathscr{L}^1} c <_{\mathscr{L}^1} a$ then consider $\mathbf{c} = (c, 1, 0, \dots, 0)$. One has $\mathbf{b} <_{\text{lex}} \mathbf{c} <_{\text{lex}} \mathbf{a}$. As in the above case, the pairs of vectors $\mathbf{b}, \mathbf{a}$ and $\mathbf{c}, \mathbf{b}$ have a common entry, so $\mathbf{b} \in A$, a contradiction. Hence, we can assume $\mathbf{a} = (\text{succ}(b), 0, \dots, 0)$ and $\mathbf{b} = (b, 1, \dots, 1)$. But then, taking into account Lemma 10(a), for the set $A' = (A \backslash \{\mathbf{a}\}) \cup \{\mathbf{b}\}$ one has $R(A') - R(A) = (R(A) - r(a) + k + r(b)) - R(A) \geqslant k - 1 \geqslant 1$, which contradicts the optimality of A. $\square$

Corollary 1. *For any $n \geqslant 1$, the lexicographic order is an MRI-order for H^n.*

The proof is obtained by applying Lemma 11 to $P = H^1$.

Theorem 1. *If $|P| \geqslant 4$, and $\mathscr{L}^1$ and $\mathscr{L}^2$ are MRI-orders for P and P^2 respectively, then either*

(a) $|P| = 4$ *and P is the N-poset, or*
(b) $|P| = 5$ *and P is the diamond poset, or*
(c) $|P| = 2^k$ *for some $k \geqslant 2$ and $\delta(P) = \delta(H^k)$.*

Proof. By Lemma 10(b), $\delta_1 = 1$. Assume $\delta_i = 1$ for $1 \leqslant i \leqslant p$. Let $A = (\{0, 1, \dots, p - 2\} \times P) \cup \{(p - 1, 0), (p - 1, 1)\} \subset P^2$. Then A is an ideal and $R(A) - R(\mathscr{F}^2(|A|)) = 1$, a contradiction. Consequently, let $i = \min\{j \geqslant 2 \mid \delta_j \neq 1\}$. By Lemma 10(b), $\delta_i \in \{0, 2\}$.

Assume $\delta_i = 0$. If $i = 2$ then P is the N-poset by Lemma 10(c). If $i > 2$ then consider the ideal $A = (\{0,1\} \times \{0,1,\ldots,i-1\}) \cup \{2,0\} \subset P^2$. One has $R(A) - R(\mathscr{F}^2(|A|)) = 1$, a contradiction.

Assume $\delta_i = 2$. Then $i \geqslant 3$ by Lemma 10(b). Now, if $4 \leqslant i < p$ then consider the ideal $A = (\{0,\ldots,i-2\} \times P) \cup \{(i-1,0),(i-1,1),(i,0),(i,1)\} \subset P^2$. One has $R(A) - R(\mathscr{F}^2(|A|)) = 1$, a contradiction. If $5 \leqslant i = p$ then consider the ideal $A = \mathscr{F}^2(6) \cup \{(2,0),(2,1),(2,2)\} \subset P^2$. One has $R(A) - R(\mathscr{F}^2(9)) = 1$, a contradiction. Therefore, $i = p = 4$, and $\delta(P) = (0,1,1,1,2)$. Hence, P is the diamond poset.

Finally, we consider the case $\delta_3 = 2$, i.e. $\delta(P) = (0,1,1,2,\delta_4,\ldots,\delta_p)$. We show by induction on t that

$$\delta_{i+2^t} = \delta_i + 1, \quad \text{for } 0 \leqslant i < 2^t \text{ and } i + 2^t \leqslant p. \tag{6}$$

For $t \leqslant 1$ this follows from the values of the first four entries of $\delta(P)$, so we assume $t \geqslant 2$. Consider the poset $S = H^2 \times P$. Since S coincides with an initial segment of size $4|P|$ in order $\mathscr{L}^2$ defined on $P \times P$, the restriction of $\mathscr{L}^2$ on S is an MRI-order for S. Note that the lexicographic order on $P \times H^2$ is isomorphic to the *reverse-lexicographic* order on $H^2 \times P$ defined as follows: $(\alpha',\beta') \in P \times H^2$ precedes $(\alpha'',\beta'') \in P \times H^2$ in the lexicographic order if and only if $(\beta',\alpha') \in H^2 \times P$ precedes $(\beta'',\alpha'') \in H^2 \times P$ in the reverse-lexicographic order. Therefore, by Lemma 11, the reverse-lexicographic order is also an MRI-order for S. Let

$$A' = \{(x,y) \in S \mid (x,y) \leqslant_{\mathscr{L}^2} (0,i)\},$$

$$A'' = \{(x,y) \in S \mid (x,y) \leqslant_{\mathscr{L}^2} (0,i+2^t)\},$$

and let B' and B'' be the initial segments of the reverse-lexicographic order on S with $|B'| = |A'|$ and $|B''| = |A''|$. Further, denote by (x',y') and (x'',y'') the maximum elements of B' and B'' in the reverse-lexicographic order, respectively. Therefore, A', A'', B', and B'' are maximum rank ideals in S, and $R(A') = R(B')$ and $R(A'') = R(B'')$. Since these ideals are initial segments of the corresponding total orders, one has

$$r_S((0,i)) = r_S((x',y')) \quad \text{and} \quad r_S((0,i+2^t)) = r_S((x'',y'')). \tag{7}$$

Since $|B'| = 2i+1$, $|B''| = 2(i+2^t)+1$ and $t \geqslant 2$, one has $x' \equiv (2i+1) \bmod 4 \equiv (2(i+2^t)+1) \bmod 4 \equiv x''$. Similarly, $y'' - y' = 2^{t-1}$. Furthermore, $y' \leqslant i/2 \leqslant 2^{t-1} - 1$, and by (7) and the inductive hypothesis, one has

$$\begin{aligned}
\delta_{i+2^t} - \delta_i &= (\delta_0 + \delta_{i+2^t}) - (\delta_0 + \delta_i) \\
&= r_S((0,i+2^t)) - r_S((0,i)) = r_S(x'',y'') - r_S(x',y') \\
&= (\delta_{x''} + \delta_{y''}) - (\delta_{x'} + \delta_{y'}) = \delta_{y''} - \delta_{y'} \\
&= \delta_{y'+2^{t-1}} - \delta_{y'} = 1,
\end{aligned}$$

so (6) is established. Hence, $\delta(P)$ matches the first p entries of $\delta(H^k)$ for $p + 1 \leqslant 2^k$. In particular, $\delta_i \geqslant 1$ for $i \geqslant 1$.

To complete the proof of the theorem, we need to show that $|P| = 2^k$ for some $k \geqslant 2$. Assume to the contrary that $|P| = 2^k + l$ for some $k \geqslant 2$ and $0 < l < 2^k$. Consider the poset $T = H^{k+1} \times P$. Since (6) implies that T coincides with an initial segment of size $2^{k+1}|P|$ in order $\mathscr{Z}^2$ defined on $P \times P$, the restriction of $\mathscr{Z}^2$ on T is an MRI-order for T. On the other hand, by Lemma 11, the reverse-lexicographic order is also an MRI-order for T.

Let A be the initial segment of T in the order $\mathscr{Z}^2$ with $|A| = 2|P| + 1$ and let B be the initial segment of T in the reverse-lexicographic order with $|B| = |A|$. Denote by (x', y') and (x'', y'') the maximum elements or A and B in the corresponding orders, respectively. Note that $(x', y') = (2, 0)$ and $(x'', y'') = (2l, 1)$. The arguments similar to those that imply (7) provide $r_T((x', y')) = r_T((x'', y''))$. One has

$$\delta_2 = r_T((x', y')) = r_T((x'', y'')) = \delta_1 + \delta_{2l}.$$

Hence, $\delta_{2l} = 0$ for some $2l \geqslant 1$, which is a contradiction. □

Note that in case (c) of Theorem 1 it is not necessarily true that $P = H^k$. However, $r(P) = r(H^k)$ and $|P_i| = |H_i^k|$ for $i = 0, \ldots, r(P)$. Furthermore, if for $i \geqslant 1$ and any $x \in P_i$ we add to P the relations $y \leqslant_P x$ for all $y \in \Delta(\mathscr{F}_i(x))$ and make a similar operation with H^k, then the resulting posets will be isomorphic (see [7]). Since the shadow function of P is the same as for H^k, such poset P is not interesting for our analysis.

4. A new class of Macaulay posets

In this section we prove that for any cartesian power P^n of the poset P, shown in Fig. 2(a) with a rank-greedy Macaulay order, the zigzag order $\mathscr{Z}^n$ is Macaulay. A subposet of P^2 formed by the vertices of the first and the second level with the order $\mathscr{Z}^2$ is shown in Fig. 2(b). Since P (and, thus, P^n for $n \geqslant 2$) is self-dual, it can be easily shown that P^2 is Macaulay.

For $\mathbf{x} = (x_1, \ldots, x_n) \in P^n(1,1)$ denote by $\pi_0(\mathbf{x})$ the vector of $P^n(1,0)$ obtained from $\mathbf{x}$ by replacing $x_1 = 1$ with 0. Furthermore, let $P_t^n(i,j) = P^n(i,j) \cap P_t^n$.

Lemma 12. *Let $A \subseteq P_t^n$ be an initial segment with $A \subseteq P_t^n(1,0) \cup P_t^n(1,1)$. Then*

$$\Delta(A) = \Delta(A(1,1)).$$

Proof. By Lemma 5 the order $\mathscr{Z}^n$ is consistent, so $A(1,0)$ and $A(1,1)$ are initial segments in $P_t^n(1,0)$ and $P_{t-1}^n(1,1)$, respectively. Clearly, $\Delta(A) \supseteq \Delta(A(1,1))$.

For the reverse inclusion, let $\mathbf{c} \in \Delta(A)$. If $\mathbf{c} \in P^n(1,1)$ then, obviously, $\mathbf{c} \in \Delta(A(1,1))$. So, we assume $\mathbf{c} \in P^n(1,0)$. Without loss of generality, we can assume $\mathbf{c}$ is the maximum element in $\Delta(A(1,0))$. Let $\mathbf{c} \in \Delta(\mathbf{a})$ for some $\mathbf{a} \in A(1,0)$. Then $\mathbf{c} <_{\mathscr{Z}^n} \mathbf{a}$ by rank-greediness (Lemma 8). Consider $\mathbf{b} \in P(1,1)$ such that $\mathbf{c} = \pi_0(\mathbf{b})$. Since $\mathbf{b}$ is the

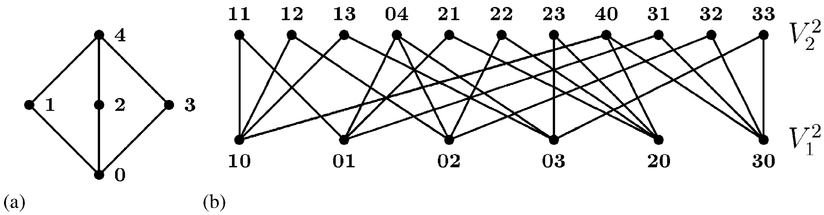

Fig. 2. (a) The diamond poset V; (b) a subposet induced by $V_2^2 \cup V_1^2$.

successor of $\mathbf{c}$ in $\mathscr{L}^n$, one has $\mathbf{c} <_{\mathscr{L}^n} \mathbf{b} <_{\mathscr{L}^n} \mathbf{a}$. Since A is an initial segment, $\mathbf{a} \in A$, and $r_{P^n}(\mathbf{a}) = r_{P^n}(\mathbf{b})$ then $\mathbf{b} \in A$. This implies $\mathbf{c} \in \Delta(A(1,1))$ and the proof is completed. $\square$

Lemma 13. *The order $\mathscr{L}^n$ on P^n satisfies the continuity property for any $n \geqslant 1$.*

Proof. We prove the lemma by induction on n. The case $n = 1$ is clear, by Fig. 2, so we assume $n \geqslant 2$. By Lemma 9 the poset P^n is rank-greedy. Let $A \subseteq P_t^n$ be an initial segment.

Case 1: Assume $A \subseteq P^n(1,0) \cup P^n(1,1)$. Let $\mathbf{a}$ be the maximum element of $A(1,1)$ and $\mathbf{b} \in \Delta(A)$ be the predecessor of $\mathbf{a}$ in order $\mathscr{L}^n$. We show that $\mathbf{b}$ is the maximum element of $\Delta(A)$. Indeed, consider $\mathbf{c} \in \Delta(A)$. If $\mathbf{c} \in P^n(1,0)$ then $\mathbf{c} <_{\mathscr{L}^n} \mathbf{b}$ follows from Lemma 12. If $\mathbf{c} \in P^n(1,1)$ then $\mathbf{c} \in \Delta(\mathbf{d})$ for some $\mathbf{d} \in A(1,1)$. By rank-greediness, $\mathbf{c} <_{\mathscr{L}^n} \mathbf{d}$ and $\mathbf{d} <_{\mathscr{L}^n} \mathbf{a}$ by the choice of $\mathbf{a}$. Finally, since $\mathbf{b}$ is the predecessor of $\mathbf{a}$ and $\mathbf{b} \neq \mathbf{d}$ then $\mathbf{c} <_{\mathscr{L}^n} \mathbf{b}$.

To complete the proof let $\mathbf{x}$ be the first element of $P_{t-1}^n \backslash \Delta(A)$. Note that $\Delta(A) \cap P^n(1,0)$ is an initial segment by Lemma 12 and, by induction $\Delta(A) \cap P^n(1,1)$ is an initial segment in the subposet $P^n(1,1)$. Since $\mathbf{b}$ is the maximum element of $\Delta(A)$, then $\mathbf{x} \in P^n(1,1)$. Since $\mathbf{x} \notin \Delta(\mathbf{a})$ and $\mathbf{y} <_{\mathscr{L}^n} \mathbf{x}$ for any $\mathbf{y} \in \Delta(\mathbf{a})$ then $\mathbf{a} <_{\mathscr{L}^n} \mathbf{x}$ by rank-greediness. This implies $\mathbf{c} <_{\mathscr{L}^n} \mathbf{x}$, so $\Delta(A)$ is an initial segment.

Case 2: Assume $A \cap P^n(1,2) \neq \emptyset$. Then $P_t^n(1,0) \cup P_t^n(1,1) \subset A$ and $P_{t-1}^n(1,0) \cup P_{t-1}^n(1,1) \subset \Delta(A)$. Let $\mathbf{x}$ be the first element not in $\Delta(A)$ and let $\mathbf{a}$ be the maximal element in $\Delta(A)$. Now, $\mathbf{a} \in \Delta(A(1,2))$ and by induction $\Delta(A(1,2))$ is an initial segment in $P^n(1,2)$. Thus $\mathbf{a} <_{\mathscr{L}^n} \mathbf{x}$.

Case 3: Assume $A \cap P^n(1,3) \neq \emptyset$. Then $P_{t-1}^n(1,0) \cup P_{t-1}^n(1,1) \cup P_{t-1}^n(1,2) \subset \Delta(A)$. Hence, if $\mathbf{a}$ is the maximal element in $\Delta(A)$ then $\mathbf{a} \in \Delta(A(1,3)) \cup \Delta(A(1,4))$. If $\mathbf{x}$ is the first element of $P_{t-1}^n \backslash \Delta(A)$ then $\mathbf{x} \in P^n(1,3) \cup P^n(1,4)$. Similar to case 1, one can show $\mathbf{a} <_{\mathscr{L}^n} \mathbf{x}$. $\square$

Now we introduce level compression. For $n \geqslant 2$ we call a subset $A \subseteq P_t^n$ *level i-compressed* if $A(i,j)$ is an initial segment of $P_{t-r(j)}^{n-1}$ in order $\mathscr{L}^{n-1}$. If A is level i-compressed for all $i = 1, \ldots, n$ then it is called *level compressed.*

Lemma 14. *Let $A \subseteq P_t^n$ and the order $\mathscr{L}^{n-1}$ be Macaulay for P^{n-1}. Then there exists a level compressed set $D \subseteq P_t^n$ such that $|D| = |A|$ and $|\Delta(D)| \leqslant |\Delta(A)|$.*

Proof. For $x \in P$ with $r(x) < r(P)$ let $\nabla(x) = \{y \in P \mid x \in \Delta(y)\}$ and let $\nabla(x) = \emptyset$ if $r(x) = r(P)$. Let $A \subseteq P_t^n$ and $B = \Delta(A)$. Then for any fixed i, $1 \leqslant i \leqslant n$, one has

$$|\Delta(A)| = \left| \bigcup_{x \in P} B(i,x) \right| \geqslant \sum_{x \in P} \max_{y \in \nabla(x)} \{|\Delta(A(i,x)) \cap P^n(i,x)|, |A(i,y)|\}. \tag{8}$$

Let C be the level i-compression of A. Then $C(i,x)$ and $\Delta(C(i,x)) \cap P^n(i,x)$ (Lemma 13) are initial segments in the $P_t^n(i,x)$ and $P_{t-1}^n(i,x)$, respectively. Therefore, the lower bound (8) for the set C is tight. Since $P^n(i,x)$ is isomorphic to the Macaulay poset P^{n-1}, then $|\Delta(A(i,x)) \cap P^n(i,x)| \geqslant |\Delta(C(i,x)) \cap P^n(i,x)|$. This implies $|\Delta(A)| \geqslant |\Delta(C)|$.

Applying level i-compression for $i = 1, \ldots, n$ sufficiently many times in the cyclic order one gets a level compressed set $D \subseteq P_t^n$ satisfying the statement. □

The main result of this section is the following theorem.

Theorem 2. *The zigzag order $\mathscr{Z}^n$ on P^n, where P is the diamond poset, is Macaulay for $n \geqslant 1$.*

Proof. We use induction on n. For $n = 1$ and 2 the proof easily follows by observing the Hasse diagrams of P and P^2 given in Fig. 2(a) and (b), respectively, so we assume $n \geqslant 3$. By Lemma 13 the order $\mathscr{Z}^n$ satisfies the continuity property, so it remains to establish the nestedness.

Let $A \subseteq P_t^n$ be an optimal set. By Lemma 14, we can assume A is level compressed. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be the last vertex of A in order $\mathscr{Z}^n$, and let $\mathbf{b} = (b_1, \ldots, b_n)$ be the first vertex of $P_t^n \backslash A$. If A is an initial segment then the theorem is true, so we assume $\mathbf{a} >_{\mathscr{Z}^n} \mathbf{b}$. Using the fact that A is compressed, we will show that either $\mathbf{b} \in A$, a contradiction, or $|\Delta(A)| \geqslant |\Delta((A \backslash \{\mathbf{a}\}) \cup \{\mathbf{b}\})|$, in which case we can swap $\mathbf{a}$ with $\mathbf{b}$ and proceed.

We can assume that $a_1 \leqslant_{\mathscr{Z}^1} 3$ and $b_1 \leqslant_{\mathscr{Z}^1} 1$. Indeed, by Lemma 3 the Macaulayness of P^n is equivalent to the Macaulayness of $(P^*)^n$. However, the roles of $\mathbf{a}$ and $\mathbf{b}$ in $(P^*)^n$ are played by $(4 - b_1, \ldots, 4 - b_n)$ and $(4 - a_1, \ldots, 4 - a_n)$, respectively. Thus, for example, the cases with $a_1 = 4$, $b_1 = 1$ and $a_1 = 3$, $b_1 = 0$ are equivalent since P^n is self dual for any $n \geqslant 1$.

Case 1: Assume $\mathbf{a} = (3, a_2, \ldots, a_n)$ and $\mathbf{b} = (1, b_2, \ldots, b_n)$. Suppose there exists an $i \geqslant 2$ such that $r(a_i) = r(b_i)$. If $a_i >_{\mathscr{Z}^1} b_i$ then, using the consistency of $\mathscr{Z}^n$ (Lemma 5),

$$\begin{aligned} \mathbf{a} = (3, a_2, \ldots, a_i, \ldots, a_n) &>_{\mathscr{Z}^n} (3, a_2, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) \\ &>_{\mathscr{Z}^n} (1, b_2, \ldots, b_n) = \mathbf{b} \in A. \end{aligned}$$

If $a_i <_{\mathscr{Z}^1} b_i$ then

$$\begin{aligned} \mathbf{a} = (3, a_2, \ldots, a_i, \ldots, a_n) &>_{\mathscr{Z}^n} (2, a_2, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) \\ &>_{\mathscr{Z}^n} (1, b_2, \ldots, b_n) = \mathbf{b} \in A. \end{aligned}$$

Any two consecutive vectors in the chains above have a common equal entry. Since $\mathbf{a} \in A$ and A is level compressed then $\mathbf{b} \in A$. A similar approach will be used in the analysis of all remaining cases. Suppose $r(a_i) \neq r(b_i)$ for all $i \geqslant 2$. There exist i and j such that $r(a_i) > r(b_i)$ and $r(a_j) < r(b_j)$. Moreover, we can assume that $r(a_i) - r(b_i) \geqslant r(b_j) - r(a_j) \geqslant 0$. Choose $a'_i \in \Delta(a_i)$. If $n \geqslant 4$ then

$$\mathbf{a} = (3, \ldots, a_i, \ldots, a_j, \ldots, a_n) >_{\mathscr{L}^n} (2, \ldots, a'_i, \ldots, b_j, \ldots, a_n)$$
$$>_{\mathscr{L}^n} (1, \ldots, b_i, \ldots, b_j, \ldots, b_n) = \mathbf{b}.$$

If $n = 3$ then $\mathbf{a} = (3, a_2, a_3)$ and $\mathbf{b} = (1, b_2, b_3)$. If $|r(a_2) - r(b_2)| = 1$, then $\mathbf{a} = (3, a_2, a_3) >_{\mathscr{L}^3} (0, a_2, a_3) >_{\mathscr{L}^3} (1, b_2, b_3) = \mathbf{b}$. One has $\mathbf{a} = (3, 4, 0) >_{\mathscr{L}^3} (3, 0, 4) >_{\mathscr{L}^3} (1, 0, 4) = \mathbf{b}$. If $|r(a_2) - r(b_2)| = 2$, then $a_2 = 0$ and $b_2 = 4$ or $a_2 = 4$ and $b_2 = 0$, and $\mathbf{a} = (3, 0, 4) >_{\mathscr{L}^3} (3, 1, 1) >_{\mathscr{L}^3} (0, 4, 1) >_{\mathscr{L}^3} (1, 4, 0) = \mathbf{b}$. In both cases, we have shown that $\mathbf{b} \in A$.

Case 2: Assume $\mathbf{a} = (2, a_2, \ldots, a_n)$ and $\mathbf{b} = (1, b_2, \ldots, b_n)$. Suppose there exists an $i \geqslant 2$ such that $r(a_i) = r(b_i) + 1$. We have

$$\mathbf{a} = (2, \ldots, a_i, \ldots, \ldots, a_n) >_{\mathscr{L}^n} (0, b_2, \ldots, a_i, \ldots, a_n) >_{\mathscr{L}^n} (1, \ldots, b_i, \ldots, b_n) = \mathbf{b},$$

since $(0, a_i) >_{\mathscr{L}^2} (1, b_i)$ for $a_i >_{\mathscr{L}^1} b_i$.

Suppose there exists an $i \geqslant 2$, such that $r(a_i) = r(b_i) + 2$ in which case $a_i = 4$ and $b_i = 0$. If $a_i = 4$ then $a_k = 4$ for $i < k \leqslant n$. Indeed, if $a_k <_{\mathscr{L}^1} 4$ for some $k > i$ then choose a'_k such that $r(a'_k) = r(a_k) + 1$. One has

$$\mathbf{a} = (2, a_2, \ldots, a_i = 4, \ldots, a_k, \ldots, a_n) >_{\mathscr{L}^n} (2, a_2, \ldots, 2, \ldots, a'_k, \ldots, a_n)$$
$$>_{\mathscr{L}^n} (0, b_2, \ldots, 2, b_{i+1}, \ldots, b_n) >_{\mathscr{L}^n} (1, b_2, \ldots, b_i = 0, \ldots, b_n) = \mathbf{b},$$

where the first inequality follows from $(4, a_k) >_{\mathscr{L}^2} (2, a'_k)$ and the last one from $(0, 2) >_{\mathscr{L}^2} (1, 0)$.

We may now assume that $\mathbf{a} = (2, \ldots, a_j, \ldots, 4, \ldots, 4)$ and $\mathbf{b} = (1, \ldots, b_j, \ldots, 0, \ldots, 0)$. Moreover, since $r_{P^n}(\mathbf{a}) = r_{P^n}(\mathbf{b})$, there exists a $j \geqslant 2$ such that $r(a_j) < r(b_j)$.

If $a_j = 0$ then, since $b_j >_{\mathscr{L}^1} 0$,

$$\mathbf{a} = (2, \ldots, a_j = 0, \ldots, 4, 4, \ldots, 4) >_{\mathscr{L}^n} (2, \ldots, a_j = 1, \ldots, 1, 4, \ldots, 4)$$
$$>_{\mathscr{L}^n} (0, \ldots, b_j, \ldots, 1, 0, \ldots, 0) >_{\mathscr{L}^n} (1, \ldots, b_j, \ldots, 0, 0, \ldots, 0) = \mathbf{b},$$

where the first inequality follows from $(0, 4) >_{\mathscr{L}^2} (1, 1)$.

If $a_j >_{\mathscr{L}^1} 0$, then, since $b_j = 4$,

$$\mathbf{a} = (2, \ldots, a_j, \ldots, 4, 4, \ldots, 4) >_{\mathscr{L}^n} (0, \ldots, 3, \ldots, 4, 4, \ldots, 4)$$
$$>_{\mathscr{L}^n} (1, \ldots, 3, \ldots, 1, 4, \ldots, 4) >_{\mathscr{L}^n} (1, \ldots, b_j = 4, \ldots, 0, 0, \ldots, 0) = \mathbf{b},$$

where the last inequality follows from $(3, 1) >_{\mathscr{L}^2} (4, 0)$.

Suppose that $r(a_i) = r(b_i)$ for all $i \geqslant 2$, which then implies $a_i \in \{1, 2\}$ and $b_i \in \{2, 3\}$. If $a_i = 2$ for all $i \geqslant 2$ then

$$\mathbf{a} = (2, 2, \ldots, 2) >_{\mathscr{L}^n} (2, 1, 3, 2, \ldots, 2) >_{\mathscr{L}^n} (1, b_2, b_3, \ldots, b_n) = \mathbf{b}.$$

Assume $a_i = 1$ for some $i \geqslant 2$. In this case $(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) \in \Delta_{\text{new}}(\mathbf{a})$, so $|\Delta_{\text{new}}(\mathbf{a})| \geqslant 1$. On the other hand, $\Delta_{\text{new}}(\mathbf{b}) = \{(0, b_2, \dots, b_n)\}$, thus $|\Delta_{\text{new}}(\mathbf{a})| \geqslant |\Delta_{\text{new}}(\mathbf{b})|$, and so $|\Delta((A \backslash \{\mathbf{a}\}) \cup \{\mathbf{b}\})| \leqslant |\Delta(A)|$, and we can swap **a** and **b**.

Case 3: Assume $\mathbf{a} = (0, a_2, \dots, a_n)$ and $\mathbf{b} = (1, b_2, \dots, b_n)$. Then $\mathbf{b}' = (0, b_2, \dots, b_n) \in \Delta(A)$ by Lemma 12. Hence, there exists a vector $\mathbf{d} = (0, b_2, \dots, d_i, \dots, b_n) \in A$, where **d** and $\mathbf{b}'$ differ in only the ith component, with $d_i >_{\mathscr{L}^1} b_i$. But now

$$\mathbf{d} = (0, b_2, \dots, d_i, \dots, b_n) > (1, b_2, \dots, b_n) = \mathbf{b},$$

since $(0, d_i) >_{\mathscr{L}^2} (1, b_i)$, so $\mathbf{b} \in A$.

Case 4: Assume $\mathbf{b} = (0, b_2, \dots, b_n)$. In this case the assumptions of Lemma 12 are satisfied for the set $\mathscr{F}^n_t(\mathbf{b})$. This implies $\Delta(A \cup \{\mathbf{b}\}) = \Delta(A)$, hence, $|\Delta((A \backslash \{\mathbf{a}\}) \cup \{\mathbf{b}\})| \leqslant |\Delta(A)|$, and we can swap **a** and **b**.

Therefore, in all cases either we have $\mathbf{b} \in A$, a contradiction, or we can swap **a** and **b** without increasing the shadow of A. After a finite number of such operations A will be transformed into an initial segment. □

5. A new construction for Macaulay posets

Let P be a Macaulay poset with associated Macaulay order $\mathcal{O}$. The poset P is called *additive* if for any $t > 0$ and any m and m' the following condition is satisfied:

$$|\Delta(\mathscr{F}_t(m'))| + |\Delta(\mathscr{F}_t(m''))| \geqslant |\Delta(\mathscr{F}_t(m' + \kappa))| + |\Delta(\mathscr{F}_t(m'' - \kappa))|, \tag{9}$$

where

$$\kappa = \kappa(m', m'') = \min\{m'', |P_t| - m'\}.$$

In other words, if we take two copies P' and P'' of P, and initial segments $\mathscr{F}_t(m') \subseteq P'_t$ and $\mathscr{F}_t(m'') \subseteq P''_t$, we should be able to move some vertices from one initial segment to the other one without increasing the sum of their shadows. This transformation is schematically shown in Fig. 3, where the initial segments are depicted in bold.

Examples of additive Macaulay posets include the Boolean lattice and the lattice of multisets, the star poset and colored complexes (see [11] for more details).

Let P be an additive Macaulay poset. First we construct a poset $P^{(k)}$ whose Hasse diagram consists of k disjoint copies of P, assuming that the tth level of $P^{(k)}$ is the union of the tth levels of the corresponding copies. This poset can be viewed as cartesian product of P with a trivial poset $T^{(k)}$ with k elements $q_1, \dots, q_k$ of rank 0, i.e. $P^{(k)} = T^{(k)} \times P$. Note that $T^{(k)}$ is a Macaulay poset with a Macaulay order $\mathscr{T}$ given by $q_1 <_{\mathscr{T}} q_2 <_{\mathscr{T}} \cdots <_{\mathscr{T}} q_k$.

Using the additivity and induction on k, it can be shown that $P^{(k)}$ is additive and Macaulay for any $k \geqslant 1$ (see [10]). The Macaulay order on $P^{(k)}$ is the lexicographic-like order: $(q', p') <_{P^{(k)}} (q'', p'')$ if and only if either $q' <_{\mathscr{T}} q''$, or if $q' = q''$ and $p' <_{\mathcal{O}} p''$.

Now we add to the relation on $P^{(k)}$ any subset of $\{(q'x, q''y) \mid q' <_{\mathscr{T}} q'' \text{ and } r_P(y) = r_P(x) + 1\}$. We denote the resulting poset by $Q^{(k)}$. This construction in the case $k = 2$

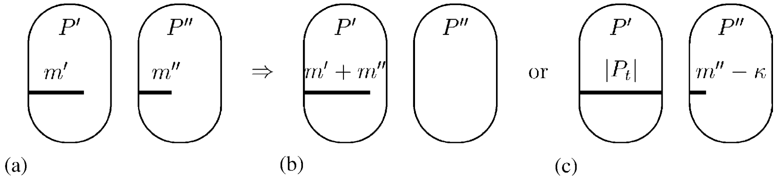

Fig. 3. The additivity of P. (a) The original configuration. The resulting sets: (b) if $\kappa = m''$ and (c) if $\kappa = |P_t| - m'$.

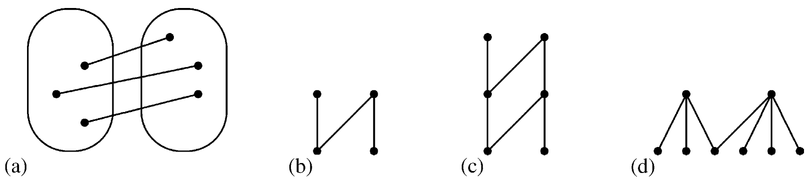

Fig. 4. (a) General construction of poset $Q^{(k)}$. (b–d) Some examples.

is shown in Fig. 4(a). The two copies of a poset P, namely $\{q_1\} \times P$ (the left one) and $\{q_2\} \times P$ (the right one), are shown by ovals together with some new edges connecting them. In particular, if the Hasse diagram of P consists of two elements connected by an edge, this construction results in the N-poset shown in Fig. 4(b). Some further examples are shown in Fig. 4(c) and (d) for the cases when the Hasse diagram of P is a chain of length 2 and a star with three rays, respectively.

Since for any subset $A \subseteq P_t^{(k)}$ one has $|\Delta_{P^{(k)}}(A)| \leqslant |\Delta_{Q^{(k)}}(A)|$ and for any initial segment of the order defined above we have an equality by the construction, $Q^{(k)}$ is Macaulay. The argument similar to the one in [10] implies that $Q^{(k)}$ is additive.

Theorem 3. *Let P be a poset and $n \geqslant 1$. If P^n is additive and Macaulay, then $(Q^{(k)})^n$ is additive and Macaulay for all $k \geqslant 1$.*

Proof. The arguments above imply the theorem is true for $n = 1$. Assume $n \geqslant 2$ and consider first the poset $(P^{(k)})^n$. One can view $(P^{(k)})^n$ as the cartesian product $(T^{(k)})^n \times P^n$. The poset $(T^{(k)})^n$ for $T^{(k)}$ specified above consists of k^n isolated elements $(z_1, \ldots, z_n)$ of rank 0 with $z_i \in \{q_1, \ldots, q_k\}$, $i = 1, \ldots, n$. In other words, $(P^{(k)})^n = (T^{(k)} \times P)^n = (T^{(k)})^n \times P^n$. We define a lexicographic-like total order $\preceq^n$ on $(P^{(k)})^n$ by setting $(z'_1, \ldots, z'_n, p') \prec^n (z''_1, \ldots, z''_n, p'')$ if and only if $(z'_1, \ldots, z'_n)$ precedes $(z''_1, \ldots, z''_n)$ in the lexicographic order or if $(z'_1, \ldots, z'_n) = (z''_1, \ldots, z''_n)$ and $p' <_{\mathcal{O}} p''$. Now let us turn to the poset $(Q^{(k)})^n$. One has

$$|\Delta_{(P^{(k)})^n}(A)| \leqslant |\Delta_{(Q^{(k)})^n}(A)|. \tag{10}$$

Let (x, y) be an edge of the Hasse diagram of $Q^{(k)} \backslash P^{(k)}$. If the vertices $(z'_1, \ldots, z'_n, x)$ and $(z''_1, \ldots, z''_n, y)$ are connected by an edge of the Hasse diagram of $(Q^{(k)})^n$, then $(z'_1, \ldots, z'_n) \neq (z''_1, \ldots, z''_n)$. Furthermore, these vectors differ in one entry only, say the ith entry. If $r(x) < r(y)$ then $z'_i <_{\mathscr{T}} z''_i$ by the construction of $Q^{(k)}$. Hence, $(z'_1, \ldots, z'_n)$ precedes $(z''_1, \ldots, z''_n)$ in the lexicographic order. By the definition of the order $\preceq^n$, if A is an initial segment of this order, then (10) becomes equality. Hence, $(Q^{(k)})^n$ is a Macaulay poset. The additivity of $(Q^{(k)})^n$ follows from the arguments presented in [10]. □

In particular, Theorem 3 in combination with known results concerning the additivity of the lattice of multisets and the star posets (see [11]), implies that any cartesian power of the posets shown in Fig. 4(b)–(d) is Macaulay.

Now we are ready to prove the following proposition that appeared in Section 2.

Proof of Lemma 6. We will first show that if $\mathscr{L}^2$ is Macaulay for P^2 then P is additive. The proof goes along the lines of the proof of Theorem 2(c) in [7]. Note that [7] deals with a different definition of the additivity, which, however, is equivalent to the one used in our paper (see [11] for details). Although the mentioned theorem is proved in [7] for the lexicographic order, its proof also works for the order $\mathscr{L}^2$. We only have to make sure that, following the notations of [7], there exist two elements $x, y \in P$ such that $(x, y) \neq (0, 1)$, $(x, y) \neq (p-1, p)$, $x <_{\mathscr{L}^1} y$ and $r(x) = r(y)$. In this case the lexicographic order on P^2 matches the order $\mathscr{L}^2$. However, if the elements x and y do not exist, then $|P_0| \leqslant 2$, $|P_{r(P)}| \leqslant 2$, and $|P_i| = 1$ for $0 < i < r(P)$. There exist just a small number of such posets if P is connected, and all of them are additive, as it is easy to verify.

Now, if P is additive and Macaulay, then we apply Proposition 6 of [7] that guarantees $\Delta(P_i) = P_{i-1}$ for any $i = 1, \ldots, r(P)$. The last condition and the fact that P is additive and Macaulay if and only if P^* is additive and Macaulay (see [11]), imply P is graded. □

6. Concluding remarks

With a little work one can also adapt the proof of Theorem 2(d,e) of [7] to show further important properties of the diamond poset such that shadow-increasing and final shadow-increasing. This properties are widely used to solve various extremal poset problems, see [6,11] for some of them.

As a byproduct of our analysis we are able to specify the class of all connected graphs G for which the order $\mathscr{L}^n$ provides nestedness in the edge-isoperimetric problem on G^n. It is shown in [3] that for any graph G and any $n \geqslant 1$ there exists a representing poset for G^n (but not versa, in general), for which the order $\mathscr{L}^n$ is an MRI-order. As soon as all representing posets are specified in Theorem 5, we can restore all represented graphs. Now, the N-poset represents no connected graph because it has two elements of zero rank (see [2,3]). Therefore, the class consists just of the series of hypercube-like graphs and a cycle of length 5 represented by the diamond poset.

We expect that our approach can also be used to specify all posets whose cartesian powers with the "star order" (see [6,11]) are Macaulay. It is shown in [2] that the cartesian powers of trees are the only connected graphs for which this order solves the edge-isoperimetric problem. What remains to be done in this direction is to specify the posets that represent no graphs. It seems that Lemmas 6–9 are of more general nature and are valid for many consistent orders (presently we know that they are valid for three different orders: the lexicographic order, the zigzag order and the simplicial order [8]). It is very interesting to specify all total orders for which they are valid.

Another candidate for adaption of our approach is the Petersen order introduced in [4]. Since it is rather close to the zigzag order, we expect a complete specification of the Macaulay posets with this order can be done with a little additional work. We also believe that the proof technique of Theorem 2 can be used to prove the Macaulayness of the posets representing the cartesian powers of the Petersen graph (see [2,4]).

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