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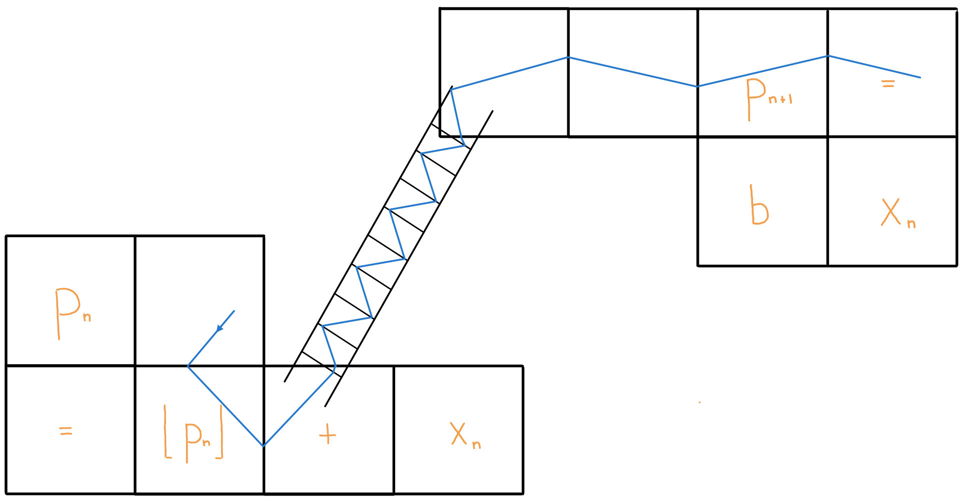

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Hexagonal Refractions

Emma Anderson and Lucy Loukes

Abstract

Hexagonal tilings yield both periodic and phase periodic orbits. Their periodic orbits can have either period 3 or period 6, meaning that it takes either 3 or 6 iterations for the orbit to return to its starting point and direction. This paper provides definitions and proofs to explain why these statements are true, and helps the reader to better understand the geometry behind orbits in a hexagonal tiling.

The idea of refractions comes from the bending of light through different mediums, such as air and water. When the straight path of a light ray reaches a boundary, that path has to be altered in some way. This idea is similar to the game of billiards, where the ball changes direction when it reaches an edge. The angle created by the change in direction does not happen by chance; it does does so in a way that can be calculated. We decided to look into these refractions to identify patterns and establish proofs, but rather than looking at a bounded tile, such as a billiard table, we decided to identify refractions through tilings of a plane.

This paper discusses our findings on refractions through the regular hexagonal tiling of a plane. Through previous research with regular rectangular and triangular tilings, we have found that there are two types of orbits: periodic and phase periodic. We will define these orbits below. These findings led us to wonder about tilings with more than 4 sides. The theorems and proofs created for the rectangular and triangular tilings helped create the proofs below for all possible orbits of a hexagonal tiling. In this hexagonal tiling, we have identified three cases, which are determined by the relationship between the edges which the orbits meet. We will identify these cases and provide justification for the theorem we present.

Definition 1. *A **tiling** is an arrangement of polygons on a plane.*

Definition 2. *A **regular tiling** is any tiling for which a polygon's edges and vertices are lined up completely. At any given vertex, the maximum number of tiles possible are meeting.*

Definition 3. *An **orbit** is a path that is piecewise linear and has the property that whenever the path intersects the edge of the tile, the outgoing trajectory is a reflection of the incoming trajectory over the edge it intersects.*

Definition 4. *A **periodic orbit** is an orbit which returns to the original starting point and direction of the path in the same tiling, shown in Figure 1.*

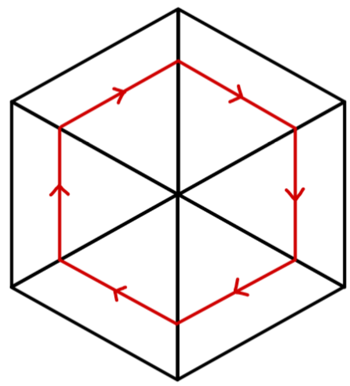

Figure 1: Periodic Orbit

Definition 5. *A* phase periodic orbit *is an orbit which returns to its starting point and direction in a different tile, shown in Figure 2.*

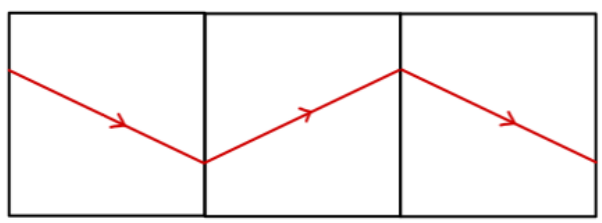

Figure 2: Phase Periodic Orbit

Theorem 6. *Consider an orbit in a regular hexagonal tiling of the plane. There are three cases:*

1. *The orbit goes to the opposite side of the tile and is phase periodic.*
2. *The orbit goes to a side that is neither opposite nor adjacent and is periodic with period 6.*
3. *The orbit goes to an adjacent side and is periodic with either period 3 or period 6.*

Proof. We will refer to the edges of a hexagonal tile using the labels shown on the diagram below. We can label any hexagon so that the first iteration of the orbit starts on side A. An orbit which goes to side D will represent case 1, an orbit which goes to side C will represent case 2, and an orbit which goes to side B will represent case 3.

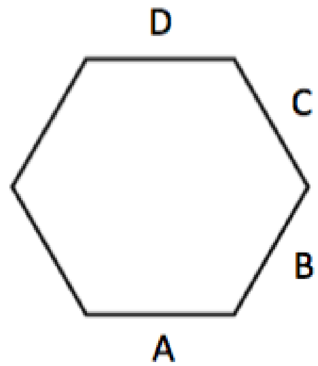

Figure 3: Labeling a Hexagonal Tile

Case 1: In this case, the orbit starts on side A and continues toward side D at an angle of θ, which is labeled in Figure 4. Because sides A and D are parallel to one another, the orbit will intersect side D at an angle of θ by the Alternate Interior Angle Theorem, which states that when two parallel lines are cut by a transversal , the resulting alternate interior angles are congruent. The first iteration of the orbit has formed a right triangle with side A and a line which can be drawn perpendicular to A and D at the point where the orbit hits D. This orbit will then reflect over D at an angle ϕ into a new tile by hypothesis. We know $\phi = \pi - \theta$ because the sum of ϕ and θ must be π. We also know that the orbit continues to intersect opposite sides of the hexagonal tiles because we can identify congruent triangles in each tile. Because we know the three angles within the right triangle are the same in each new tile, we know the side lengths remain the same as well, so we know the orbit continues to intersect tiles at the same location. Once again, we can reflect the orbit over the edge of the tile, this time at an angle of $\pi - \phi$, which is equal to $\pi - (\pi - \theta)$, or θ. This takes the orbit to its original starting point and direction in a new tile, which begins the second iteration of the sequence we started with and forms a phase periodic orbit. We have therefore demonstrated that an orbit which intersects opposite sides of a hexagonal tile will always yield a phase periodic orbit.

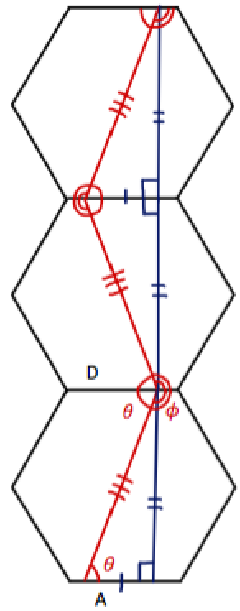

Figure 4: Case 1 Orbit

Case 2: In this case, the orbit starts on side A and continues toward side C. We can draw lines through sides A and C which intersect at an angle of $\frac{\pi}{3}$. In Figure 5 below, these lines are green. Our orbit will meet A at an angle θ, and it will meet C at an angle of ϕ. The green lines going through A and C form a triangle with the first iteration of the orbit. The angles within any triangle must add up to π, so our three angles, $\frac{\pi}{3}, \theta$, and ϕ, must do this. We can solve for ϕ:

$$\pi = \frac{\pi}{3} + \theta + \phi$$

$$\phi = \pi - \left(\frac{\pi}{3} + \theta\right)$$

We have identified a triangle whose angles are $\frac{\pi}{3}, \theta, and \phi$. When the first iteration of our orbit reflects over C, it does so at an angle of ϕ, and because we know the angle between the two green lines is $\frac{\pi}{3}$, the remaining angle must be θ. This second triangle is congruent to the first, so its side lengths are the same. From this, we know the second iteration of the orbit will meet sides A and C of its new tile. This process continues until the sixth reflection brings the orbit back to its original position on A, meeting the tile at an angle of θ. When the orbit reflects over A, it will follow the exact path it started in. This demonstrates that an orbit will always be periodic with period 6 when it intersects two edges of a hexagonal tile which are neither adjacent nor opposite to each other.

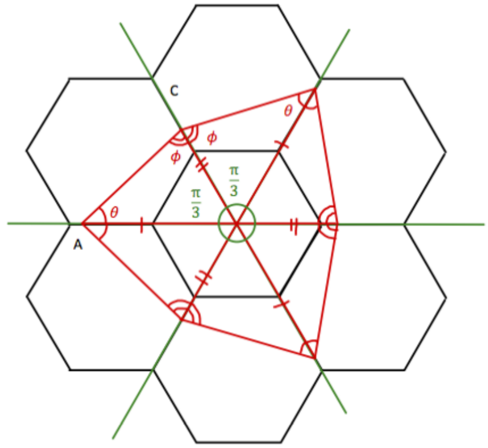

Figure 5: Case 2 Orbit

Case 3: Within this case, we have two subcases, 3a and 3b:

3a: $\theta = \frac{\pi}{6}$
In this case, the orbit will meet side A at an angle of $\frac{\pi}{6}$. Given that the angle between A and B is $\frac{2\pi}{3}$, we can solve for the angle ϕ at which the orbit meets side B. We do this by setting all three angles equal to π because we know they must form a triangle:

$$\pi = \frac{2\pi}{3} + \frac{\pi}{6} + \phi$$

$$\phi = \pi - \left(\frac{2\pi}{3} + \frac{\pi}{6}\right)$$

$$\phi = \pi - \frac{5\pi}{6}$$

$$\phi = \frac{\pi}{6}$$

Because $\phi = \frac{\pi}{6}$, the orbit forms an isosceles triangle with two sides it intersects, which we can see in Figure 6. This means that the point of intersection is the same on both edges because the side lengths are equal. By our hypothesis, the orbit will reflect over B at an angle of $\frac{\pi}{6}$, meeting the next edge of the new tile at this same angle. This process repeats only three times before the orbit returns to its starting point and angle. This forms the only periodic orbit with period 3 in a hexagonal tiling, because $\theta = \frac{\pi}{6}$ is the only angle for which $\theta = \phi$. With this, we have proven that an orbit which intersects two adjacent edges of a hexagonal tile and has an angle of $\frac{\pi}{6}$ will always be periodic with period 3.

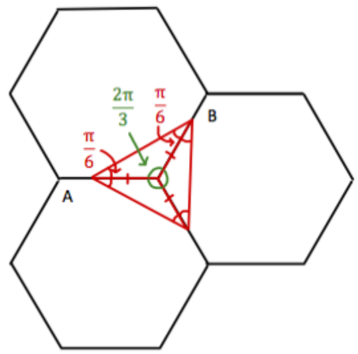

Figure 6: Case 3a Orbit

3b: $\theta \neq \frac{\pi}{6}$
In this case, the orbit starts on side A and goes toward side B. Because the adjacent edges of a hexagon meet at an angle of $\frac{2\pi}{3}$, the orbit will leave A at an angle of θ and meet B at an angle of ϕ, forming a triangle with the two edges. We can solve for ϕ:

$$\pi = \frac{2\pi}{3} + \theta + \phi$$

$$\phi = \pi - \left(\frac{2\pi}{3} + \theta\right)$$

Because $\theta \neq \frac{\pi}{6}$ in this case, we know $\theta \neq \phi$, so the triangle these angles form is scalene for any given value of θ. When the first iteration of the orbit is reflected over B at an angle of ϕ, it meets the edge of the next tile at an angle of θ, forming a scalene triangle which is congruent to the first. With the next reflection, the orbit returns to side A of the first tile, but at an angle of ϕ and at a different distance from the origin. Because of this, the orbit must complete three more reflections before it meets side A at the same point and angle from which it started. This pattern is demonstrated more clearly in Figure 7 below. With this, we have proven that an orbit which intersects two adjacent edges of a hexagonal tile and does not have an angle of $\frac{\pi}{6}$ will always be periodic with period 6.

□

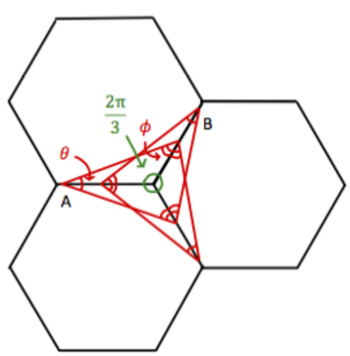

Figure 7: Case 3b Orbit

Modeling Chutes & Ladders Using Markov Chains

Jamie Woodworth

Abstract

Chutes & Ladders is a popular children's game of chance where players take turns spinning a fair spinner, advancing their position until they reach the end of the board. Each board has a configuration of chutes and ladders, which regress and advance the position of the player respectively. Any board configuration can be represented using an absorbing Markov Chain. This is created by adding matrices that represent chutes and ladders to the matrix representing the blank board. This paper presents the general form for a blank board given its size and spinner regions, and the general form for a chute or ladder given its position and length.

1 Introduction

Chutes & Ladders is a board game where players race to move their game piece from the start to end of the board. A traditional board has 100 spaces labelled from 1 to 100, as seen in Fig. 1. The game piece starts on square 1 in the lower left corner, and the end space is square 100 in the upper left corner. Notice that when the game piece gets to the end of the first row, the game piece is moved vertically upwards to the next row and moves leftwards on the new row. This pattern of movement continues for each new row, causing the path of the game piece to wind upwards on the board.

A fair spinner is used to determine the amount of spaces the player moves the game piece for a given turn. If a spinner had 4 equal regions and the player spun a 3 on the spinner, the player would advance the game piece by 3 spaces on the board. The game ends when the game piece lands *exactly* on the end space. If the player spins a value greater than the amount of spaces it takes for the game piece to land on the end space, the game piece remains on its current space. If the game piece is 3 spaces away from the end space and the player spins a value of 4, then the game piece remains in its current position.

As can be seen in Fig. 1, Chutes & Ladders boards are populated with features called chutes and ladders which regress and advance the position of the game piece respectively. Features connect two spaces on the board, which will be referred to as the position space and the destination space. When the game piece lands on the position space of a feature, it is forced to move directly to the destination space of the feature. In Fig. 1, space 28 is the position space for a ladder, and the destination space is space 84. Ladders have a destination space ahead of its position space, and chutes have a destination space behind its position space.

Definition 1. *A* ***board*** *is an ordered triple (s, n, F), where s is the amount of spaces on the board labelled sequentially from 1 to s, n is the amount of regions of the spinner labelled from 1 to n, and F is the set of features on the board. When $F = \emptyset$, the board is called a "blank board".*

Definition 2. *A* ***feature*** *is an ordered pair of non-zero integers (p, l) where $0 < p \leq s$ and $-p \leq l \leq (s - p)$. p is the position space of the feature, l is the length of the feature, and $p + l$ is the destination space. If $l < 0$, the feature is called a "chute" and if $l > 0$, the feature is called a "ladder".*

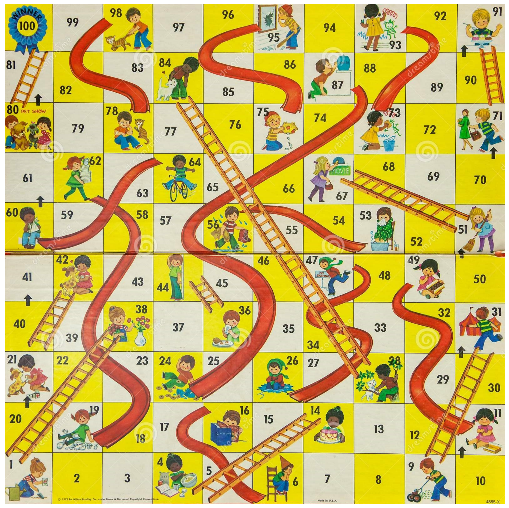

Figure 1: A traditional Chutes & Ladders game board.

A game of Chutes & Ladders can be represented by a system of states and transitions between those states. Each position on the board is a state, and the player transitions the game piece between those states by spinning the spinner and moving the game piece accordingly. Once the game piece reaches the end space, the game piece is unable to move to another space since the game is over. Since each move is probabilistic due to the randomness of the spinner, a board is an example of a Markov Chain.

Definition 3. *A* ***Markov Chain*** *is a system that describes probabilistic transitions between states. If there exists one or more states that it is impossible to transition into a different state from, it is called an "absorbing" Markov Chain.*

Definition 4. *A* ***transition matrix*** *is a mathematical representation of a Markov Chain in matrix form. The i, j entry of the transition matrix is the probability of transitioning from state j to state i.*

As an example, here is the transition matrix for a $(4, 2, \emptyset)$ board:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 1 \end{bmatrix}$$

The first column of this matrix contains the probabilities of moving from first space of the board to each space on the board. From the first space, the probability of moving to space 2 is 0.5 and the probability of moving to space 3 is 0.5, which corresponds to spinning a value of 1 or 2 respectively. The third column represents the transitions from third space of the board. If a 1 is spun, the game piece moves to the end space and if a 2 is spun, the game piece remains on the third space. The final column is the end space, which is the absorbing state of the matrix. The game piece cannot be moved to any other space, so it will remain on the end space with a probability of 1. Each column of the transition matrix must add to 1, since the game piece will always exist on some space on the board.

2 Generalization of Blank Boards

Theorem 5. *The transition matrix for a $(s, n, \emptyset)$ board is:*

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{n} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ddots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{2}{n} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{3}{n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{n-1}{n} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 1 \end{bmatrix}$$

Proof. The transition matrix for a board of size s must be an $s \times s$ matrix because it must describe the probability of transitioning from each state to each state, with a total of s states. The entries must be in multiples of $\frac{1}{n}$, since the probabilities of spinning each value on our spinner is $\frac{1}{n}$. The blank board contains no features, so the probability of moving from space j to a space $i < j$ and the probability of moving from space j to a space $i > j + n$ is 0. This is because the game piece cannot move backwards without a chute and cannot move forwards more spaces than the spinner has regions without a ladder.

When $j \leq s - n$, there are enough spaces ahead of the game piece such that it can advance freely. The probability of staying on the current space is 0, and there is a probability of $\frac{1}{n}$ of moving to each of the subsequent n spaces. When $s - n < j < s$, there are only $s - j$ spaces ahead of the game piece. For example, when $j = s - n + 1$, there are only $n - 1$ spaces ahead of the game piece. This means that if the player spins a value of n, the game piece must remain in its current position. For each space closer to the end the game piece is, there is one additional value on the spinner that forces the game piece to remain in its current position. When $s - n < j < s$, the probability of moving to each subsequent space is $\frac{1}{n}$ and the probability of remaining on the current space is $\frac{j-s+n}{n}$. □

3 Generalization of Features

Theorem 6. *The transition matrix for a (s, n, F) board where $F = \{(p, l)\}$ can be represented by the sum of the transition matrix for a $(s, n, \emptyset)$ board and a (p, l) feature matrix.*

Proof. The presence of a feature effectively removes its position space from the board. The game piece moves immediately to the destination space upon landing on the position space, so it is impossible to start or end a turn on the position space. Landing on the position space is equivalent to landing on the destination space. The probability of moving to the destination space is equal to the probability of spinning a value that moves the game piece to the position space plus the probability of spinning a value that moves the game piece to the destination space. By taking a blank board and subtracting all probability of beginning or ending a turn on the position space, and adding the probability of ending a turn on the position space to the probability of ending a turn on the destination space, the transition matrix for a $(s, n, \{(p, l)\})$ board emerges. □

Theorem 7. *The feature matrix of a (p, l) feature on a $(s, n, \emptyset)$ board is:*

$$\begin{bmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -\frac{1}{n} & \cdots & -\frac{1}{n} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{1}{n} & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{1}{n} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}$$

Proof. The feature makes it impossible to start a turn in position p, so all entries in the p^{th} column of the blank board must be negated by the feature matrix. The same is true for the entries in the p^{th} row of the blank board, because it is impossible to end a turn on position p. The probabilities subtracted from the p^{th} row will be added the the $(p + l)^{th}$ row, since the probability of landing on $(p + l)$, the destination space, has increased by the probability of landing on space p. □

4 Conclusions

We have shown that given the desired size and amount of spinner regions, the transition matrix of a blank board can be constructed. We have also shown that given the desired position and length of a feature on this blank board, a feature matrix can be constructed. By adding the feature matrix to the transition matrix of the blank board, a transition matrix for a board containing the specified feature can be constructed. Any number of features can be added to a blank board in this fashion, provided that none of the position or destination spaces overlap. Using this method, any conceivable transition matrix for a Chutes & Ladders board can be constructed.

Since any Chutes & Ladders board can now be represented as a matrix, it can be analyzed using linear algebra techniques. This will allow future investigation into properties such as the average number of turns it takes to win a board, and how the configuration of features on a board impacts this expected value.

Counting Divisions of $2 \times n$ Board

Jacob Brown

Abstract

In this report we explore the number of ways to divide an $m \times n$ rectangular grid, cutting along the grid lines, into k pieces. Durham and Richmond [1] previously identified that the number of divisions of a $2 \times n$ board into two pieces is $2n^2 - n$. We extend this result by offering an alternative proof and by counting the number of divisions into three and four pieces. We achieve this by proving a recursive relationship and then using computational methods to find polynomials that fit the number of divisions with respect to n. We also provide an algorithm for counting the number of divisions of an arbitrary board into k pieces.

1 Introduction

Consider an $n \times m$ rectangle composed of 1×1 squares. Cutting only along the edges between squares, how many ways are there to divide this rectangular board into two pieces? Three pieces? Four? What about an arbitrary number of pieces?

Here, we answer these questions for a $2 \times n$ rectangular board. In this section we introduce definitions and notation, as well as some basic results. In Section 2, we offer an alternative proof of the result by Durham and Richmond [1], who previously identified that the number of divisions of a $2 \times n$ rectangular board into two pieces is $2n^2 - n$. In Section 3, we prove a recursive relationship for the divisions of the $2 \times n$ board into an arbitrary number of pieces. In Section 4, we give an algorithm that counts the number of divisions of an arbitrary board into an arbitrary number of pieces. Finally, in Section 5, we prove explicit formulas for the number of divisions of the $2 \times n$ board into three and four pieces.

We begin with the following definitions and remarks:

Definition 1. *An $n \times m$ **board** is a rectangle consisting of 1×1 **squares**. The board consists of n squares vertically and m squares horizontally. Two squares are **connected** if and only if they share an edge.*

Definition 2. *A **cut** is an action that separates two connected squares along their shared edge.*

Definition 3. *A **division** of a board into k **pieces** is a sequence of cuts that result in the board being separated into k connected components. See Figure 1 for an example division of a 3×5 board into 3 pieces.*

Definition 4. *$d_k^m(n)$ denotes the number of divisions of an $n \times m$ board into k pieces. For example, $d_3^3(5)$ is the number of divisions of a 3×5 board into 3 pieces.*

Definition 5. *$s_k^2(n)$ denotes the number of divisions of an $2 \times n$ board into k pieces where the two rightmost squares are separated by a cut along their shared edge. Likewise, $t_k^2(n)$ denotes the number of divisions of an $2 \times n$ board into k pieces where the two rightmost squares are not separated by a cut along their shared edge. Equivalently, this means the squares end up in the different pieces / the same piece, respectively.*

Lemma 6. $d_k^2(n) = s_k^2(n) + t_k^2(n)$

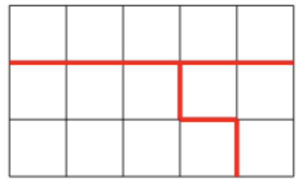

Figure 1: A division of a 3×5 board into 3 pieces. The red lines each represent a cut. Note that the long horizontal line is five cuts connected together.

Proof. The set of divisions is partitioned by whether or not a cut exists along the edge between the rightmost squares. □

Remark 7. *$d_0^m(n) = 0$ for all m, n, as there is no way to divide the board into 0 pieces.*

Remark 8. *$d_1^m(n) = 1$ and $d_{mn}^m(n) = 1$ for all m, n, as there is only one way to divide the board into one piece (leave it fully intact), and only one way to divide it into mn pieces (cut each square into its own piece).*

2 Closed Form Solution for Counting Divisions into Two Pieces

Durham and Richmond previously showed that the number of divisions of the $2 \times n$ board into two pieces is given by $2n^2 - n$. Our argument makes use of induction and recursion, and these arguments will be utilized in a similar manner in the following sections. A key insight to our argument is that the divisions of a $2 \times n$ board are embedded within the divisions of a $2 \times (n+1)$ board, and the divisions of this "sub-board" (consisting of all but the rightmost column) correspond in a predictable manner with the divisions of the overall board.

Theorem 9. *The number of divisions of a $2 \times n$ board into two pieces is $d_2^2(n) = n(2n-1)$.*

Proof. Our proof is by induction on n.

By Remark 8, $d_2^2(1) = 1 = (1)(2(1) - 1)$.

For the induction, assume for a positive integer j that $d_2^2(j) = j(2j-1)$. Each division of the $2 \times j$ sub-board has some number of corresponding divisions in the $2 \times (j+1)$ board. Note that the sub-board's division may be into only one piece. Color the rightmost vertically-connected squares of the $2 \times (j+1)$ board blue and the rightmost vertically-connected squares of the $2 \times j$ sub-board green (see Figure 2). To obtain a division into two pieces on the $2 \times (j+1)$ board, there are two cases to consider. First, there's the case where the $2 \times j$ sub-board is divided into only one piece. In this case, there are three ways to obtain a division of the $2 \times (j+1)$ board into two pieces. Two divisions can be obtained by cutting off one of the blue squares and attaching the other to the $2 \times j$ sub-board, and one division is obtained by cutting off both blue squares into their own piece (see Figure 3).

The remaining divisions can be obtained by dividing the $2 \times j$ sub-board into two pieces and adding the blue squares onto them. Recall from Lemma 6 that the number of divisions of the $2 \times j$ sub-board is partitioned by whether or not the green squares separate. Furthermore, for each division, there will be some number of ways to add the blue squares to obtain a division of the $2 \times (j+1)$ board. Therefore, we can write:

$$d_2^2(j+1) = 3 + At_2^2(j) + Bs_2^2(j) \tag{1}$$

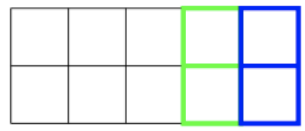

Figure 2: Coloring argument that is used in the proof of Theorem 9.

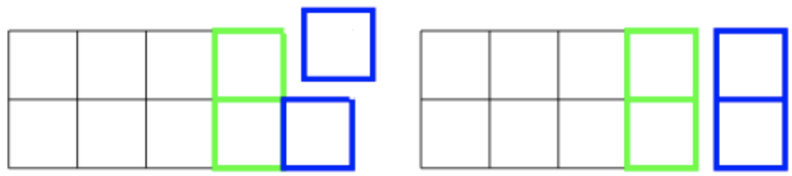

Figure 3: The three ways for the blue squares to be added to the board when the $2 \times j$ sub-board remains fully intact. The case on the left accounts for two ways, depending on which blue square becomes its own piece.

For some positive integers A, B. To find A, note that when the green squares do not separate, the blue squares must remain connected and attach to the green squares, or else the board will be divided into more than two pieces. Therefore, for each division of the $2 \times j$ sub-board where the green squares do not separate, there is one corresponding division in the $2 \times (j+1)$ board. Thus, $A = 1$

When the green squares do separate, there are three possible sub-cases: i) both blue squares attach to the top green square, ii) both blue squares attach to the bottom green square, or iii) the blue squares separate and attach one to each green square (see Figure 4). Therefore, for each division of the $2 \times j$ board where the green squares do separate, there are 3 corresponding divisions in the $2 \times (j+1)$ board, thus $B = 3$.

Rewriting Equation 1 using Lemma 6:

$$d_2^2(j+1) = 3 + t_2^2(j) + 3s_2^2(j) = 3 + 2s_2^2(j) + d_2^2(j) \tag{2}$$

To count $s_2^2(j)$, note that the only divisions of the $2 \times j$ sub-board that cause the green squares

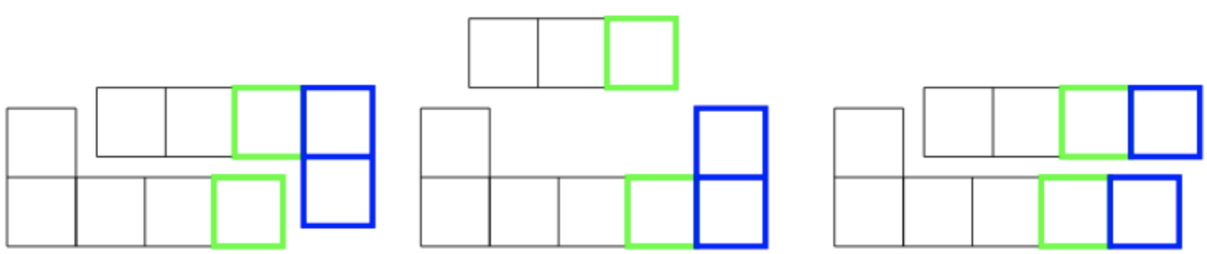

Figure 4: The three ways to add the blue squares to a division of the $2 \times j$ sub-board where the green squares separate.

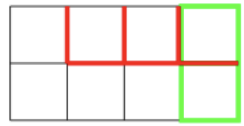

Figure 5: Each vertical red line is a possible edge to place a vertical cut that, when paired with the horizontal cut to the right edge of the board, removes the upper right corner of the board and results in the green squares becoming separated.

to separate are the division that divides the board along the horizontal middle and the divisions that remove only the corner of the board. To remove only a corner of the board, place a vertical cut along any of the $j-1$ vertical edges that are contained in the interior of the board (ie, those edges that are not on the outer boundary). Then, make a horizontal cut along the horizontal middle of the board in the direction of the corner to be removed (See Figure 5). Thus, there are $j-1$ ways to remove one corner from the board, and because there are two corners being considered, the total number of divisions that separate the green squares is

$$s_2^2(j) = 2(j-1) + 1 = 2j - 1 \tag{3}$$

Substituting this into Equation 2, and using the inductive assumption:

$$d_2^2(j+1) = 3 + 2(2j-1) + j(2j-1) = 2j^2 + 3j + 1 = (j+1)(2(j+1)-1) \tag{4}$$

Which completes the induction. □

3 Recursive Relationship for Counting Divisions into Arbitrary Number of Pieces

Given the result of the previous section, a natural next step is to count the number of divisions into more than 2 pieces. The arguments used in the proof of Theorem 9 can be generalized to count the divisions into an arbitrary number of pieces, we show that here. In Section 5, we use the results in this section to find the number of divisions of the board into 3 and 4 pieces.

To count the divisions of a $2 \times n$ board into k pieces, a recursive relationship can be created. The recursion is built upon the divisions of the nested sub-board within the overall board, as in the proof of Theorem 9.

Theorem 10. *The number of divisions of a $2 \times n$ board into k pieces satisfies the following recursion:*

$$d_k^2(n+1) = d_{k-2}^2(n) + 3d_{k-1}^2(n) + d_k^2(n) + 2s_k^2(n) \tag{5}$$

Proof. Consider a $2 \times n$ board, and then add two squares onto the rightmost squares. To distinguish these squares from the rest of the board, color them blue and color the rightmost squares of the $2 \times n$ sub-board green (see Figure 2). To find the divisions of the $2 \times (n+1)$ board into k pieces, consider the ways that the blue squares can be attached to the pieces of the divisions of the $2 \times n$ board.

The blue squares can be used to add zero, one, or two pieces to each division of the $2 \times n$ sub-board. Therefore, we only need to consider the recursive relationships of the divisions of the $2 \times n$ sub-board into $k-2$, $k-1$ and k pieces.

If the $2 \times n$ board is divided into $k-2$ pieces, simply add each blue square as a separate piece. Because there is only one way to add the blue squares to each division of the $2 \times n$ board into $k-2$ pieces, there will be a term of $1 \cdot d^2_{k-2}(n)$ in the recursion.

If the $2 \times n$ board is divided into $k-1$ pieces, then for each division there are 3 ways to add the two blue squares in order to obtain a division of the $2 \times (n+1)$ board into k pieces. These additions follow the same argument summarized in Figure 3. Therefore, for each division of the $2 \times n$ board into $k-1$ pieces, there are 3 corresponding divisions of the $2 \times (n+1)$ board into k pieces, so a term of $3 \cdot d^2_{k-1}(n)$ will appear in the recursion.

If the $2 \times n$ board is divided into k pieces, then the number of corresponding divisions in the $2 \times (n+1)$ board into k pieces depends on whether the rightmost squares of the $2 \times n$ board remain connected of if they separate. For each division where they remain connected, there is only one way to add the blue squares to obtain a corresponding division in the $2 \times (n+1)$ board, that is, adding the blue squares to the piece containing both of the rightmost squares of the $2 \times n$ board. Therefore there will be a term of $1 \cdot t^2_k(n)$ in the recursion.

For each division where the rightmost squares of the $2 \times n$ board separate, there are 3 ways to add the blue squares to obtain a corresponding division in the $2 \times (n+1)$ board. This follows the same argument for counting the additional divisions when the green squares separate as in the proof of Theorem 9. Thus for each division of the $2 \times n$ board into k pieces where the rightmost pieces separate, there are 3 corresponding divisions of the $2 \times (n+1)$ board into k pieces. This means there is a term of $3s^2_k(n)$ in the recursion.

Because there can be no terms involving divisions of the $2 \times n$ board into fewer than $k-2$ and greater than k pieces, these are all the terms of the recursion, so:

$$d^2_k(n+1) = d^2_{k-2}(n) + 3d^2_{k-1}(n) + t^2_k(n) + 3s^2_k(n) \tag{6}$$

Substitute using Lemma 6 to obtain the final recursion above. □

As a check of this result, we show that the result of Theorem 9 satisfies this recursion, using Remarks 7 and 8 and the expression for $s^2_2(n)$ found in the proof of Theorem 9:

$$d^2_0(n) = 0, \quad d^2_1(n) = 1, \quad d^2_2(n) = n(2n-1), \quad s^2_2(n) = 2n-1$$

$$\begin{aligned} d^2_2(n+1) &= 0 + 3(1) + n(2n-1) + 2(2n-1) \\ &= 3 + 2n^2 - n + 4n - 2 = 2n^2 + 3n + 1 \\ &= (n+1)(2n+1) \\ &= (n+1)(2(n+1)-1) \end{aligned}$$

This recursion is only of use when $s^2_k(n)$ can be calculated. Fortunately, $s^2_k(n)$ can also be computed recursively, in terms that do not involve $d^2_k(n)$.

Theorem 11. *The number of divisions of a $2 \times n$ board into k pieces where the rightmost squares separate satisfies the following recursion:*

$$s^2_k(n+1) = d^2_{k-2}(n) + 2d^2_{k-1}(n) + s^2_k(n) \tag{7}$$

Proof. Using the same blue squares from the proof of Theorem 10, finding these divisions simply involves counting the number of divisions of the $2 \times (n+1)$ board where the blue squares separate. The blue squares separate in the $d^2_{k-2}(n)$ term, in two of the cases of the $d^2_{k-1}(n)$ term, none of the cases of the $t^2_k(n)$ term, and one of the cases of the $s^2_k(n)$ term, so the recursion is just the sum of these cases. □

4 Algorithm for Counting Divisions of $n \times m$ Board into k Pieces

While $d^m_k(n)$ has directly proven explicit formulas for small values for m and k, computational methods are much more useful to collect data about greater values. We used those data to fit polynomials that predict the number of divisions of the $2 \times n$ board into more than two pieces; we prove those formulas satisfy the recursion of Theorem 10 in the next section.

Our algorithm involves encoding the board as a graph, where the vertices represent the squares, and two vertices are adjacent if and only if their squares are connected in the board. A cut corresponds to removing an edge from the graph, and every valid division of the board corresponds to removing some set of edges from the graph to produce some number of connected components. The algorithm involves checking all possible combinations of edges to be removed, in other words, it iterates through the power set of the edge set of the graph, and finds the number of connected components for each graph after the edge removal.

However, not every sequence of cuts results in a valid division. For a division to be valid, any two squares that are separated by a cut must end up in different pieces of the division. In the algorithm, this means that if two vertices are adjacent before the edge removal and are not adjacent after the edge removal, the two vertices must not be in the same connected component of the resulting graph.

Using this check for validity, the following algorithm returns the number of divisions of an $n \times m$ board into 1 through mn pieces. While the algorithm is computationally expensive, it is exhaustive. Therefore no divisions are missed by the algorithm and the divisions into all numbers of pieces are returned by the algorithm at once.

See the appendix for pseudocode of the algorithm.

5 Explicit Formulas for Divisions of $2 \times n$ Board into Three and Four Pieces

Using the recursive relations from Section 3, we can find explicit formulas for $d^2_3(n)$ and $d^2_4(n)$. First, closed forms of $s^2_3(n)$ and $s^2_4(n)$ are found. All of the formulas in this section were found by generating data using the algorithm in the previous section, and then fitting a polynomial until the highest-ordered terms had coefficients equal to 0. Because an n degree polynomial is completely determined by $n+1$ known points, once the higher-order terms dropped out, it means the fitting polynomial fits all the points exactly.

Lemma 12. $s^2_3(n) = \dfrac{4}{3}n^3 - 3n^2 + \dfrac{8}{3}n - 1$

Proof. For the base case, there are 0 divisions of the 2×1 board into 3 pieces, meaning there must be 0 divisions where the rightmost pieces separate, so $s^2_3(1) = 0$. The closed form solution has a root at $n = 1$, so this satisfies the base case. For the inductive assumption, assume the above closed form solution holds for n. Then, from Theorem 11, Theorem 9, and Remark 8:

$$\begin{aligned}
s_3^2(n+1) &= d_1^2(n) + 2d_2^2(n) + \frac{4}{3}n^3 - 3n^2 + \frac{8}{3}n - 1 \\
&= (1) + 2n(2n-1) + \frac{4}{3}n^3 - 3n^2 + \frac{8}{3}n - 1 \\
&= \frac{4}{3}n^3 + n^2 + \frac{2}{3}n \\
&= \frac{4}{3}(n+1)^3 - 3(n+1)^2 + \frac{8}{3}(n+1) - 1
\end{aligned}$$

□

With this result, a closed form solution for $d_3^2(n)$ can be found.

Theorem 13. $d_3^2(n) = \frac{2}{3}n^4 - \frac{4}{3}n^3 + \frac{11}{6}n^2 - \frac{13}{6}n + 1$

Proof. For the base case, there are 0 divisions of the 2×1 board into 3 pieces, so $d_3^2(1) = 0$, and the closed form solution has a root at $n = 1$, which satisfies the base case. For the inductive assumption, assume the above closed form solution holds for n. Then substituting using Lemma 12:

$$\begin{aligned}
d_3^2(n+1) &= d_1^2(n) + 3d_2^2(n) + d_3^2(n) + 2s_3^2(n) \\
&= (1) + 3n(2n-1) + \frac{2}{3}n^4 - \frac{4}{3}n^3 + \frac{11}{6}n^2 - \frac{13}{6}n + 1 + 2\left(\frac{4}{3}n^3 - 3n^2 + \frac{8}{3}n - 1\right) \\
&= \frac{2}{3}n^4 + \frac{4}{3}n^3 + \frac{11}{6}n^2 + \frac{1}{6}n \\
&= \frac{2}{3}(n+1)^4 - \frac{4}{3}(n+1)^3 + \frac{11}{6}(n+1)^2 - \frac{13}{6}(n+1) + 1
\end{aligned}$$

□

Again using the recursive relations from Section 3, we can find an explicit formula for $d_4^2(n)$. First, a closed form of $s_4^2(n)$ is found.

Lemma 14. $s_4^2(n) = \frac{4}{15}n^5 - \frac{4}{3}n^4 + \frac{11}{3}n^3 - \frac{37}{6}n^2 + \frac{167}{30}n - 2$

Proof. The same argument applies for the base case as in the divisions into three pieces, so $s_4^2(1) = 0$. For the inductive assumption, assume the above closed form solution holds for n. Then, from Theorem 11:

$$\begin{aligned}
s_4^2(n+1) &= d_2^2(n) + 2d_3^2(n) + \frac{4}{15}n^5 - \frac{4}{3}n^4 + \frac{11}{3}n^3 - \frac{37}{6}n^2 + \frac{167}{30}n - 2 \\
&= n(2n-1) + 2\left(\frac{2}{3}n^4 - \frac{4}{3}n^3 + \frac{11}{6}n^2 - \frac{13}{6}n + 1\right) + \frac{4}{15}n^5 - \frac{4}{3}n^4 + \frac{11}{3}n^3 - \frac{37}{6}n^2 + \frac{167}{30}n - 2 \\
&= \frac{4}{15}n^5 + n^3 - \frac{1}{2}n^2 + \frac{7}{30}n \\
&= \frac{4}{15}(n+1)^5 - \frac{4}{3}(n+1)^4 + \frac{11}{3}(n+1)^3 - \frac{37}{6}(n+1)^2 + \frac{167}{30}(n+1) - 2
\end{aligned}$$

□

With this result, a closed form solution for $d_4^2(n)$ can be found.

Theorem 15. $d_4^2(n) = \frac{4}{45}n^6 - \frac{2}{5}n^5 + \frac{25}{18}n^4 - \frac{7}{2}n^3 + \frac{226}{45}n^2 - \frac{18}{5}n + 1$

Proof. The same argument applies for the base case as in the divisions into three pieces, so $d_4^2(1) = 0$. For the inductive assumption, assume the above closed form solution holds for n. Then substituting using Lemma 14:

$$\begin{aligned}
d_4^2(n+1) &= d_2^2(n) + 3d_3^2(n) + d_4^2(n) + 2s_4^2(n) \\
&= n(2n-1) + 3\left(\frac{2}{3}n^4 - \frac{4}{3}n^3 + \frac{11}{6}n^2 - \frac{13}{6}n + 1\right) \\
&+ \frac{4}{45}n^6 - \frac{2}{5}n^5 + \frac{25}{18}n^4 - \frac{7}{2}n^3 + \frac{226}{45}n^2 - \frac{18}{5}n + 1 \\
&+ 2\left(\frac{4}{15}n^5 - \frac{4}{3}n^4 + \frac{11}{3}n^3 - \frac{37}{6}n^2 + \frac{167}{30}n - 2\right) \\
&= \frac{4}{45}n^6 + \frac{2}{15}n^5 + \frac{13}{18}n^4 - \frac{1}{6}n^3 + \frac{17}{90}n^2 + \frac{1}{30}n \\
&= \frac{4}{45}(n+1)^6 - \frac{2}{5}(n+1)^5 + \frac{25}{18}(n+1)^4 - \frac{7}{2}(n+1)^3 + \frac{226}{45}(n+1)^2 - \frac{18}{5}(n+1) + 1
\end{aligned}$$

□

6 Future Work

Obviously, finding explicit formulas for the divisions of the $2 \times n$ into five pieces and more is of immediate interest. The data provided by the algorithm in Section 4 limited our analysis to a 2×9 board. The algorithm is computationally expensive, each additional column of the 2-width board multiplies the runtime by a factor of 8 (3 new edges are added for each additional column). Refining the algorithm, or simply using a more powerful computer, would allow more data to be generated and polynomials to be fit.

An interesting observation is that the degree of the polynomials appears to follow a predictable pattern. The number of divisions of the $2 \times n$ board into k pieces is given by a polynomial of degree $2(k-1)$, up to our observations for $k \leq 4$. Whether or not this pattern continues is unknown.

A Algorithm for Counting Divisions

Algorithm 1: Counting Divisions of $m \times n$ Board.

```
input  : m, n
initialize DivisionCountDictionary;
initialize Graph;
for i ∈ range(m * n) do
    add vertex i to Graph;
    // Each vertex is represented as an integer from 0 to mn − 1.
    // Along each row, the vertices are incremented up by 1.
    // Along each column, the vertices are incremented up by n.
end
for vertex1 ∈ Graph do
    for vertex2 > vertex1 ∈ Graph do
        if vertex1 == vertex2 − 1 and  mod (vertex2, n) ≢ 0 then
            add edge (vertex1,vertex2) to Graph;
            // Adds edge between adjacent vertices in row.
            // mod function ensures edge is not added between vertices on different rows.
        end
        if vertex1 == vertex2 − n then
            add edge (vertex1,vertex2) to Graph;
            // Adds edge between adjacent vertices in column.
        end
    end
end
set SavedGraph ← Graph;
// Save graph before edges are removed to reset graph for next iteration.
set PowerSet ← power set of Graph.EdgeSet;
// The power set contains all possible combinations of edges to remove.
for set ∈ PowerSet do
    for edge ∈ set do
        remove edge from Graph;
    end
    set Components ← connected components of Graph;
    set Valid ← True;
    for edge ∈ set do
        set vertex1 ← edge [0];
        set vertex2 ← edge [1];
        // Get the vertices connected by the edge.
        // Example:  edge (4,5) would store 4 in vertex1 and 5 in vertex2
        for comp ∈ Components do
            if vertex1 ∈ comp and vertex2 ∈ comp then
                set Valid ← False;
            end
        end
    end
    if Valid then
        set NumberComponents ← number of connected components of Graph;
        add 1 to DivisionCountDictionary [NumberComponents ];
        // Adds one to the count corresponding to the number of connected components.
        // Example:  DivisionCountDictionary [3] would be the entry of 3 components.
    end
    set Graph ← SavedGraph;
end
print(DivisionCountDictionary);
```

References

[1] Samuel Durham and Tom Richmond. “Connected Subsets of an $n \times 2$ Rectangle”. In: *College Mathematics Journal* 50.5 (2019), pp. 1–11.

Exploration of Variations on an Algorithm for Recording Real Numbers

Carl Fortna and Sainabou Jallow

Abstract

Real numbers can be written and analyzed in different bases. The base b representation of a number yields a sequence of non-negative integers whose dynamical properties we study below. In this project, we consider positive integer bases. We use algorithm b as a tool to investigate the periodicity of numbers.

The analysis and discussion of periodicity or a lack thereof for number patterns are carried out. We formulate equations to generate periodic numbers in any given base and we state several ideas of how to use each point to generate periodic numbers in other bases. For a given base, b, a real number is produced that is periodic in that base with period n, where n is greater than 1. From there, we compare the period and the characteristics of each orbit. We prove that if a number is periodic in an integer base then it is either periodic or eventually periodic in any integer base. We make connections between the periodicity of a number, 1/a, where a is an integer, in base 2 and in other bases, specifically showing that the number is eventually periodic in bases that are not relatively prime to a and are periodic in all other bases. Python was used to look at a fraction represented as a decimal in an arbitrary whole number base. The conjectures and proofs in this paper are based around periodicity and the length of periods for different whole number base representations. What happens when we compare periodicity of a number in different whole number bases? Can we create equations to define numbers in a specific base and period? What are some relationships between periodicity and bases?

1 Algorithm b

This writes any number in base b.
Let p be a real number. Set $p_0 = p$ and for any $n \geq 0$,

1. Write $p_n = q_n + x_n$ where $q_n = \lfloor p_n \rfloor$ and $x_n = p_n - \lfloor p_n \rfloor$. x_n is the remainder when p_n is divided by 1, so we also write that $x_n = p_n \bmod 1$. Notice that $x_n \epsilon [0, 1]$.
2. Set $p_{n+1} = bx_n$.
3. Repeat steps 1 and 2. record the numbers q_n you get in step 1 as $[p]_b = q_0.q_1q_2q_3\dots$

Example:
$b = 10$

$$p_0 = \frac{1}{3} = 0 + \frac{1}{3}$$

so $q_0 = 0$ and $x_0 = \frac{1}{3}$

$$p_1 = 10 * x_0 = \frac{10}{3} = 3 + \frac{1}{3}$$

so $q_1 = 3$ and $x_1 = \frac{1}{3}$

$$p_2 = 10 * x_1 = \frac{10}{3} = 3 + \frac{1}{3}$$

so $q_2 = 3$ and $x_2 = \frac{1}{3}$
x_n will remain $\frac{1}{3}$ so the following q_n will be 3 for all n. Therefore, $[\frac{1}{3}]_{10} = 0.\overline{33}$
From this point forward we use $\overset{b}{\Rightarrow} q_n, x_n$ to denote one step of the algorithm where b is the number base we are writing p_n in.

Definition 1. *A real number, p, is* ***periodic in base b*** *if when running algorithm b there exists an n such that $x_n = x_0$.*

Lemma 2. *A real number, p , is periodic in base b iff $[p]_b = 0.\overline{q_1 q_2 \ldots q_n}$ for some n.*

Proof. Forthcoming. □

Definition 3. *A real number, p, is* ***eventually periodic*** *if there exists an n such that $x_{i+n} = x_i$ where $i \neq 0$.*

Lemma 4. *A real number, p, is eventually periodic in base b iff $[p]_b = 0.q_1 q_2 \ldots \overline{q_i q_{i+1} \ldots q_{i+n}}$ for some n.*

Proof. Forthcoming. □

Definition 5. *The* ***period*** *of a number, n, is defined as the smallest length of the string of digits that are repeating.*

2 Exploring Periodicity

We tested different conjectures and proved some of them to obtain the results discussed below. While testing the conjectures, we use a python script that automates any algorithm, b, for any number of decimal places for any number $\frac{1}{a}$ (Appendix 1).

Theorem 6. *For any number $\frac{1}{a}$, $[\frac{1}{a}]_b$ is either eventually periodic or periodic for $a, b, n \in \mathbb{Z}_+$*

Proof. We begin with $[\frac{1}{a}]_b$ and by the definition of algorithm b, we show that

$$\frac{1}{a} \Rightarrow 0, \frac{1}{a} \overset{*b}{\Longrightarrow} q_1, x_1 \overset{*b}{\Longrightarrow} q_2, x_2 \overset{*b}{\Longrightarrow} q_3, x_3 \ldots$$

Since x_n is in the interval $[0, 1]$ and can be written in the form $\frac{m}{a}$, where m is an integer, the numerator is always less than the denominator and there are only a different possibilities, therefore, our algorithm shows that x_n will always loop back around and hence, it will be eventually periodic or periodic.

□

Theorem 7. *The following fraction $[\frac{1}{a}]_b = 0.\overline{0000 \ldots (b-1)}$, with $n-1$ zeros before $b-1$, if and only if $\frac{1}{a} = \frac{1}{b^{n-1} + b^{n-2} + b^{n-3} \cdots + 1}$, where $a, b, n \in \mathbb{Z}_+$*

Proof. We will begin by looking at the forward direction of this proof. Let $[\frac{1}{a}]_b = 0.\overline{0000\ldots(b-1)}$ with $n-1$ zeros before $b-1$
Using algorithm one:

$$\frac{1}{a} \Rightarrow 0, \frac{1}{a} \xRightarrow{*b} 0, \frac{b}{a} \xRightarrow{*b} 0, \frac{b^2}{a} \ldots \xRightarrow{*b} (b-1), \frac{1}{a}$$

Because there are $n-1$ zeros before $b-1$, we see that

$$\frac{b^n}{a} = (b-1) + \frac{1}{a}$$

and therefore,

$$\frac{b^n - 1}{a} = b - 1$$

Now, using algebra

$$a = \frac{b^n - 1}{b - 1}$$

We can factor a b out of the numerator and see that

$$a = \frac{b * (b^{n-1} - \frac{1}{b})}{b - 1}$$

By algebra, this can be moved into the bottom of the fraction in the denominator

$$a = \frac{b^{n-1} - \frac{1}{b}}{\frac{b-1}{b}}$$

Again, we can factor out a b^{n-1} from the numerator and with algebra show that

$$a = \frac{(b^{n-1}) * (1 - (\frac{1}{b})^n)}{1 - \frac{1}{b}}$$

which has the form of a finite geometric series where, first term= b^{n-1} and the common ratio= $\frac{1}{b}$, and hence,

$$a = b^{n-1} + b^{n-2} + b^{n-3} + \cdots + 1$$

Now, we will look at the backward direction of this proof. Let $\frac{1}{a} = \frac{1}{b^{n-1}+b^{n-2}+b^{n-3}+\cdots+1}$ When looking at the denominator of this fraction

$$a = b^{n-1} + b^{n-2} + b^{n-3} + \cdots + 1$$

we can see the equation for the sum of a finite geometric, with the form

$$a = \frac{b^{n-1} * (1 - (\frac{1}{b})^n)}{1 - \frac{1}{b}}$$

Now, using algebra we show that

$$a = \frac{b^{n-1} - \frac{1}{b}}{\frac{b-1}{b}}$$

Again, we can factor out a b from the denominator and move it to numerator

$$a = \frac{b * (b^{n-1} - \frac{1}{b})}{b - 1}$$

$$a = \frac{b^n - 1}{b - 1}$$

Now, we multiply both sides of the equation by $(b - 1)$ and show

$$a * (b - 1) = b^n - 1$$

We can add a 1 to both sides of the equation and then divide by a to see that

$$a * (b - 1) + 1 = b^n$$

$$(b - 1) + \frac{1}{a} = \frac{b^n}{a}$$

This shows that $b^n \bmod a = \frac{1}{a}$ which proves that:

$$\frac{1}{a} \Rightarrow 0, \frac{1}{a} \xRightarrow{*b} 0, \frac{b}{a} \xRightarrow{*b} 0, \frac{b^2}{a} \ldots \xRightarrow{*b} (b - 1), \frac{1}{a}$$

Hence, $[\frac{1}{a}]_b = 0.\overline{0000 \ldots (b - 1)}$ with $n - 1$ zeros before $b - 1$. $\square$

Lemma 8. *For all integers g, h, and x, $g * (hmod(x))mod(x) = g * hmod(x)$*

Proof. To prove this lemma, we must show that both sides are equal. Let's begin with the right-hand side, let $g = x * Q_1 + R_1$ and $h = x * Q_2 + R_2$, where Q_1 and Q_2 are integers, and $0 \leq R_i < x$

$$g * h = (xQ_1 + R_1)(xQ_2 + R_2)$$

After distribution, we show that

$$g * h = x^2 Q_1 Q_2 + xQ_1 R_2 + xQ_2 R_1 + R_1 R_2$$

Taking $mod(x)$ of both sides,

$$g * hmod(x) = R_1 R_2 mod(x)$$

because by the definition of modular arithmetic, we can see that any part of the equation multiplied by an x is divisible by x, so we are left with the remainder
Now, we will show that the left-hand side equals the right-side. Begin by looking at the part of the equation in the parenthesis

$$hmod(x) = R_2$$

Therefore,

$$g * (hmod(x))mod(x) = gR_2mod(x)$$

Again, we see that

$$gR_2mod(x) = ((xQ_1 + R_1)R_2)mod(x)$$

We can distribute the R_2 and take $mod(x)$ of both sides.

$$(xQ_1R_2 + R_1R_2)mod(x) = R_1R_2mod(x) = g * hmod(x)$$

Therefore, we have shown that $g * (hmod(x))mod(x) = g * hmod(x)$. $\square$

Theorem 9. *If $[\frac{1}{a}]_b$ is periodic with period b, then $[\frac{1}{a}]_{b^b}$ has period 1.*

Proof. We begin by looking at $[\frac{1}{a}]_b = 0.\overline{q_1q_2\dots q_b}$
By algorithm b, we see that

$$p_0 = \frac{1}{a} = 0 + \frac{1}{a}$$
$$p_1 = b * \frac{1}{a} = q_1 + \frac{b}{a}mod(1)$$

From lemma 5,

$$p_2 = b * \left(\frac{b}{a}mod(1)\right) = q_2 + \left(\frac{b^2}{a}mod(1)\right)$$
$$p_3 = b * \left(\frac{b^2}{a}mod(1)\right) = q_3 + \left(\frac{b^3}{a}mod(1)\right)$$

Therefore,

$$p_b = b * \left(\frac{b^{b-1}}{a}mod(1)\right) = q_b + \left(\frac{b^b}{a}mod(1)\right)$$

Because $[\frac{1}{a}]_b$ has period b,

$$\frac{b^b}{a}mod(1) = \frac{1}{a}$$

We will now look at the right-side of the theorem. For $[\frac{1}{a}]_{b^b}$,

$$p_0 = \frac{1}{a} = 0 + \frac{1}{a}$$
$$p_1 = b^b * \frac{1}{a} = y_1 + \left(\frac{b^b}{a}mod(1)\right)$$

where y_1 represents $\lfloor p_n \rfloor$
From the proof above,

$$\frac{b^b}{a}mod(1) = \frac{1}{a}$$

Therefore,

$$[\frac{1}{a}]_{b^b} = 0.\overline{y}_1$$

This proves that $[\frac{1}{a}]_{b^b}$ has period 1 if $[\frac{1}{a}]_b$ is periodic with period b. $\square$

A Appendix

Automation

```
import sympy as sy
sy.init_printing()
from sympy import sin, cos, pi
from sympy import *
import numpy as np
import matplotlib.pyplot as plt
import math

it = pi
base = 2
n = 20
place = 0
while place < n:
    a=floor(it)
    it=base*(it-a)
    place+=1
    print (a,end=" ")
```