Periodic Billiard Orbits on Surfaces of Revolution

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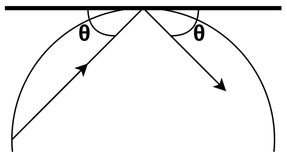
Background

Billiards

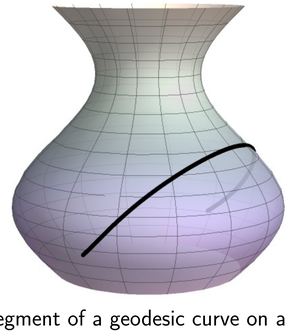
- A *billiard table* B is is a closed region whose boundary is an oriented piecewise C^2 curve
- Tables naturally admit a continuous dynamical system inspired by the game of billiards
- Particles move in straight lines and collide with the billiard boundary so that their angle of incidence is equal to the angle of reflection
- The *orbit* of a particle is its path following the billiard rules, beginning from some initial position and angle



Angle of incidence equals angle of reflection with respect to the line tangent to the billiard table

Surfaces of Revolution

- We study billiard tables placed on *surfaces of revolution*, surfaces created by rotating C^∞ curves in the xz-plane around the $z-$axis.
- On curved surfaces, orbits travel alongs paths known as *geodesics*, which generalize the notion of a constant-velocity straight line path
- Surfaces of revolution have *parallels* and *meridians*, mirroring lines of latitude and longitude on the Earth

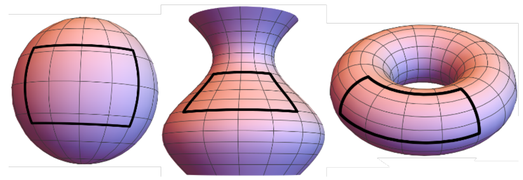


A segment of a geodesic curve on a vase

Motivation: Periodicity on Rectangles

Definition. A $w \times h$ *rectangle* is a billiard board on a surface of revolution defined by segments of parallels on the top and bottom of coordinate width w and segments of meridians on the left and right of coordinate length h.

Definition. A billiard orbit is *periodic* if it returns to the initial point with the inital angle after n bounces. We call n the *period* of the orbit.



Rectangles on a sphere, a vase, and a torus

Rectangle Periodicity Theorem

In a $w \times h$ rectangular billiard board, orbit γ is periodic in B if and only if there exist natural numbers $i, j \in \mathbb{N}$ such that

$$i\rho_\gamma \equiv 2jw \mod 2\pi.$$

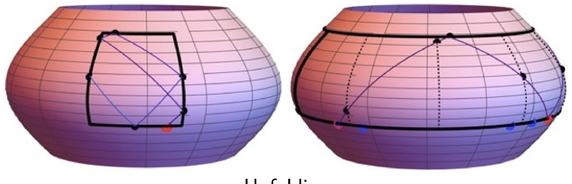
Then the period of γ is given by

$$Per(\gamma) = \begin{cases} 2i + 2j & \text{if } \gamma \text{ reaches the top of the rectangle,} \\ i + 2j & \text{otherwise,} \end{cases}$$

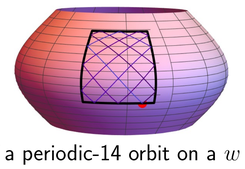
where i and j are the smallest such that the above is satisfied.

Unfoldings and Rotation

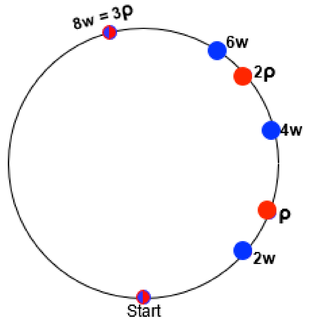
An orbit γ on a rectangle can be unfolded over meridian sides. We can quantify how fast an unfolded orbit rotates around the z-axis using a rotation number ρ_γ.



Unfolding



An example of a periodic-14 orbit on a $w \times h$ rectangle.



A top down view of ρ and w in an unfolding of a period-14 orbit, where $i = 3,\ j = 4$ satisfies the periodicity conditions.

Periodicity occurs when an unfolded orbit intersects an even reflection of P. This occurs when a multiple of the rotation number lines up with an even multiple of the width.

Corollaries

Corollary. An equatorial triangle on a sphere is an $h \times w$ rectangle with base at the equator and top at the north pole. All orbits on such triangles are periodic if

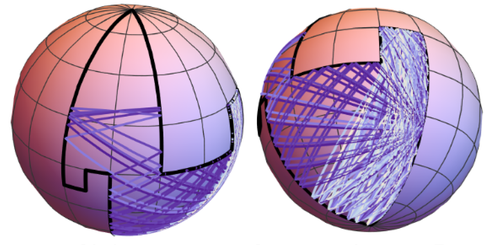
$$w = \frac{i}{2j}\pi.$$

Otherwise, no orbits are periodic.

Proof: On a sphere, all billiard orbits have $\rho = \pi$. Inserting this into our formula shows that every orbit γ on such a table is periodic with $\mathrm{Per}(\gamma) = i + 2j$.

Corollary. There exist billiard tables on the sphere that are *unilluminable*: if the walls of the table were mirrored, any candle placed inside it would fail to light up the whole room, no matter where it was placed.

Proof: This follows from constructing the billiard board out of two equatorial triangles with $w = \frac{1}{2j}\pi$ and two small $w \times h$ rectangles, all connected along their bases.



Two views of light rays coming from a single point P in an unilluminable room. No matter where P is positioned in the lower triangle, the square region off the upper triangle is never illuminated.

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