2019 Math Exploration Day
Team Competition
Answers and Solutions

1. $2/5$
2. 24
3. $1 + \sqrt{2}$
4. 13
5. $1010/1009$
6. $(6, -2)$
7. 13
8. 132
9. $680{,}403$
10. $4/11$
11. 15
12. $2\sqrt{3}$
13. 900
14. $5/9$
15. 29.0625, or $29\frac{1}{16}$
16. 4
17. 15
18. 58
19. 2
20. $3/4$

1. You use part of your summer earnings to buy several equally-priced gifts for your friends. If you use one fifth of your money to buy one third of the gifts, what fraction of your money remains after buying all of the gifts?

$\frac{2}{5}$

Let M be the summer money that you earned and let G be the cost of the gifts. Then,

$$\frac{1}{5}M = \frac{1}{3}G \implies G = \frac{3}{5}M \implies \frac{2}{5}M \text{ remains.}$$

2. A student drove a compact car 240 miles from home to the beach, averaging 30 miles per gallon. On the trip back to home, the student drove his friend's SUV, which averaged 20 miles per gallon. What was the average gas mileage, in miles per gallon, for the student's round trip?

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On the trip to the beach, the student used $240/30 = 8$ gallons and on the return trip used $240/20 = 12$ gallons. So, the student used 20 gallons to travel the full 480 miles. Thus, the average gas mileage is 24 miles per gallon.

3. A circle of radius 1 is surrounded by 4 circles of radius R as below. What is the exact value of R?
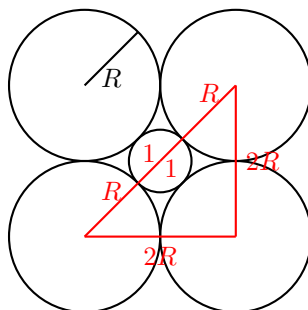
$1 + \sqrt{2}$

$$(2R)^2 + (2R)^2 = (2R+2)^2 \implies 8R^2 = 4R^2 + 8R + 4 \implies R^2 - 2R - 1 = 0 \implies R = \frac{2 \pm \sqrt{4+4}}{2}$$

Thus, $R = 1 + \sqrt{2}$.

4. Define the operation $\star$ by $x \star y = (x+y)y$. What is the value of $(6 \star 7) - (7 \star 6)$?

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$$(6 \star 7) - (7 \star 6) = (6+7)7 - (7+6)6 = 13.$$

5. Express as a fraction in lowest terms: $\dfrac{1 + 2 + 3 + 4 + \cdots + 2019}{1 + 2 + 3 + 4 + \cdots + 2018}$

$$\boxed{1010/1009}$$

$$\frac{1+2+3+4+\cdots+2019}{1+2+3+4+\cdots+2018} = \frac{(2019\cdot 2020)/2}{(2018\cdot 2019)/2} = \frac{2020}{2018} = \frac{1010}{1009}.$$

6. The point $(6,2)$ is reflected across the line $x - y = 0$ and then rotated $90°$ clockwise about the origin. What are the coordinates of the transformed point?

$$\boxed{(6,\text{-}2)}$$

The reflection moves $(6,2)$ to $(2,6)$, and then the rotation moves it to $(6,\text{-}2)$.

7. For how many real numbers x is $\sqrt{169 - \sqrt{x}}$ a positive integer?

$$\boxed{13}$$

Set $\sqrt{169 - \sqrt{x}} = m$, where m is an integer. Then,

$$169 - \sqrt{x} = m^2 \implies \sqrt{x} = 169 - m^2 \implies x = (169 - m^2)^2.$$

But,

$$0 \le 169 - \sqrt{x} \le 169 \implies 0 \le m^2 \le 169 \implies 0 \le m \le 13.$$

Thus, the 13 positive integers are $1, 2, 3, \ldots, 13$.

8. The area of a right triangle is 330 and its hypotenuse has length 61. What is the perimeter of the triangle?

$$\boxed{132}$$

Let b and h be the base and height, respectively, of the triangle. Then,

$$bh = 660 \text{ and } b^2 + h^2 = 61^2 \implies b^2 + \left(\frac{660}{b}\right)^2 = 61^2 \implies b^4 - 3721b^2 + 435600 \implies b = 11 \text{ or } b = 60.$$

If $b = 11$ then $h = 60$, and if $b = 60$ then $h = 11$. In either case, the perimeter is $11 + 60 + 61 = 132$.

9. What is the sum of every multiple of 3 from the list of whole numbers from 1 to 2019, inclusive?

$$\boxed{680{,}403}$$

There are $2019/3 = 673$ multiples of 3 between 1 and 2019: $3, 6, 9, \cdots, 2019$. Factoring 3 out from each multiple, the sum of these numbers is

$$3 \cdot (1 + 2 + 3 + \cdots + 673) = 3 \cdot \frac{673 \cdot 674}{2} = 680,403.$$

10. The local animal shelter adopts out cats and dogs. Adoptions are 10% higher than last year. The number of dog adoptions increased by 5% and the number of cat adoptions increased by 20%. What fraction of this year's adoptions were cats?

$$\boxed{4/11}$$

Let A be the number of adoptions last year, D be the number of dog adoptions last year, and C be

the number of cat adoptions last year. So, $A = C + D$. From the given information about this year's adoptions,

$$1.2C + 1.05D = 1.1A \implies 1.2C + 1.05D = 1.1C + 1.1D \implies 0.05D = 0.1C \implies D = 2C.$$

So, the fraction of this year's adoptions that were cats is

$$\frac{1.2C}{1.1A} = \frac{1.2C}{1.1D + 1.1C} = \frac{1.2C}{2.2C + 1.1C} = \frac{1.2}{3.3} = \frac{4}{11}$$

11. Consider the line through the points $(1, 7)$ and $(6, 42)$. How many points on this line have the property that both coordinates are integers and the points lie inside the circle $x^2 + y^2 = 2500$?

$$\boxed{15}$$

The line through the points $(1, 7)$ and $(6, 42)$ is $y = 7x$. The points on this line and inside the circle are:
$$(0, 0), \pm(1, 7), \pm(2, 14), \pm(3, 21), \pm(4, 28), \pm(5, 35), \pm(6, 42), \pm(7, 49)$$

12. Equilateral triangle $\triangle ABC$ has side length 4, D is the midpoint of $\overline{BC}$, and C is the midpoint of $\overline{AE}$. What is the area of the triangle $\triangle CED$.
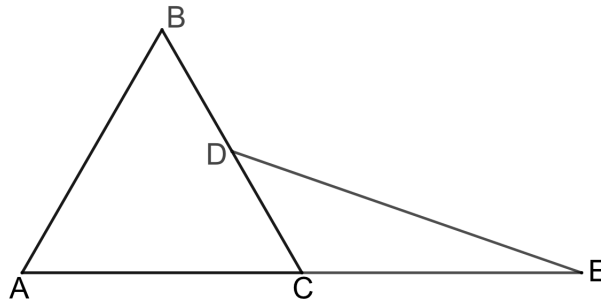
$$\boxed{2\sqrt{3}}$$

Draw a line segment from D to F on $\overline{AC}$, perpendicular to $\overline{AC}$. Then $\triangle CDF$ is a 30-60-90 triangle with base 1 and height $\sqrt{3}$; hence, it has area $\sqrt{3}/2$. Next, $\triangle FED$ has base 5 and height $\sqrt{3}$; hence, its area is $5\sqrt{3}/2$. Putting it all together,

$$\text{area}(\triangle CED) = \text{area}(\triangle FED) - \text{area}(\triangle CDF) = 5\sqrt{3}/2 - \sqrt{3}/2 = 2\sqrt{3}.$$

13. How many palindromes are there between 10,000 and 100,000? A palindrome is a number that reads the same forward and backward, such as 121.

$$\boxed{900}$$

The palindromes in this range are of the form $abcba$. There are 9 choices for a and 10 choices for each of b and c. So, there are a total of $9 \cdot 10 \cdot 10 = 900$ palindromes.

14. Two different numbers are chosen at random from the set $\{3, 8, 9, 14, 17, 22, 31, 34, 40, 51\}$. What is the probability that the sum of the two numbers is odd?

$\boxed{5/9}$

The only way to add two numbers and get an odd sum is to choose an even-odd pair. The probability of choosing an even then an odd is $(1/2)(5/9) = 5/18$. Similarly, the probability of choosing an odd then an even is $(1/2)(5/9) = 5/18$. So, the probability of an even sum is $5/18 + 5/18 = 5/9$.

15. You drop a rubber ball from a height of 10 meters. Each time the ball strikes the ground it bounces straight back up exactly half the distance of the height it just fell. If your friend catches the ball at the peak after the fifth bounce, what is the total distance that the ball travelled?

$\boxed{29\frac{1}{16}m}$
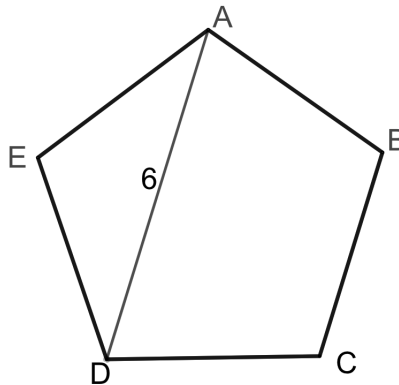
Summing the drop/bounce distances we have

$$10 + 2(5) + 2(5/2) + 2(5/4) + 2(5/8) + 5/16 = 29\frac{1}{16}.$$

16. Solve for x: $\log(\log 5) + \log x = \log(\log 625)$

$\boxed{4}$

$$
\begin{aligned}
\log(\log 5) + \log x = \log(\log 625) \implies & \log x = \log(\log 625) - \log(\log 5) \\
\implies & \log x = \log\left(\frac{\log 625}{\log 5}\right) \\
\implies & \log x = \log\left(\frac{\log 5^4}{\log 5}\right) \\
\implies & \log x = \log\left(\frac{4\log 5}{\log 5}\right) \\
\implies & \log x = \log 4 \\
\implies & x = 4
\end{aligned}
$$

17. Given regular pentagon $ABCDE$, connect the midpoints of consecutive sides with line segments to create another pentagon inside the original. If the diagonal AD has length 6, what is the perimeter of the interior pentagon?

Notice that the diagonal AD is part of $\triangle ADE$. When the interior pentagon is created, one of the sides is created by joining the midpoints of AE and ED. This side is parallel to diagonal AD and by the midpoint theorem for triangles, the length of the pentagon side must be $1/2$ of the length of AD. Thus, the side length of the pentagon is 3 and the perimeter is 15.

18. The first term in my sequence is 16. The next term in the sequence is the sum of the square of each digit in the previous terms; that is, the second term is $1^2 + 6^2$. Each of the remaining terms in my sequence is the sum of the squares of the digits of the previous term. What is the 2019th number in my sequence?

First, note that

$$x_1 = 16, x_2 = 37, x_3 = 58, x_4 = 89, x_5 = 145, x_6 = 42, x_7 = 20, x_8 = 4, x_9 = 16 = x_1.$$

So, the sequence repeats with a period of 8. Note that $2019 = 252 \cdot 8 + 3$, so that

$$x_{2019} = x_{3+252\cdot 8} = x_3 = 58.$$

19. What digit (1-9) must go in the highlighted cell of the following Sudoku puzzle?
(Reminder: In a Sudoku puzzle, the digits 1-9 must appear exactly once in each row, in each column, and in each 3×3 block indicated by thick lines.)

		4			6		2	
		7	8			9	1	
						3		8
	1	8	3			2		
3		?	7	8	9			1
					1		6	
8		3				5		
	4	5			3	6		
	2	6	5			1		

20. A point P is chosen at random from the rectangular region with vertices $(0,0), (2,0), (2,1)$, and $(0,1)$. What is the probability that P is closer to the origin than it is to the point $(2,2)$?

Consider the figure below. The oblique line represents the set of points that are equidistant from the origin and from (2,2). So, the probability that P is closer to the origin than it is to the point (2,2) is

ratio of the area of the non-shaded region within the rectangle to the area of the entire rectangle. The shaded region has area 1/2. So, the probability is

$$\frac{2 - 1/2}{2} = \frac{3}{4}.$$