

# 2022 Math Exploration Day

## Team Competition

### Solution

1. Megan worked out on a treadmill for 15 minutes, walking at 4.2 miles per hour for part of the time and running at 5.4 miles per hour for the remainder of the time. If she traveled a total of 1.3 miles, how long did she spend walking?

2.5 minutes

Let  $w$  be the walking time (in hours) and let  $r$  be the running time (in hours). The information in the problem leads to

$$4.2w + 5.4r = 1.3 \implies 4.2w + 5.4(.25 - w) = 1.3 \implies w = \frac{1}{24} \text{ hour} = 2.5 \text{ minutes}$$

2. You roll two standard, six-sided dice and you subtract the smaller number from the larger number. If you roll the same number on each die, it doesn't matter the order in which you subtract. What is the most likely difference to occur?

1

3. Olivia raises chickens and cows. She has 54 animals with a total of 122 legs. How many chickens does she have?

47

Let  $x$  be the number of chickens and let  $y$  be the number of cows. Then, we have the system  $x + y = 54$  and  $2x + 4y = 122$ , which has solution  $x = 47$  and  $y = 7$ .

4. Triangle  $ABC$  has coordinates  $A = (4, 0)$ ,  $B = (0, 4)$ , and  $C$  is a point on the line  $x + y = 10$ . What is the area of triangle  $ABC$ ?

12

The location of  $C$  along the line does not change the area, so set  $C = (10, 0)$ . The resulting triangle has area  $(1/2)(4)(10 - 4) = 12$ .

5. Define the operation  $\odot$  by  $x \odot y = y^2 - x$ . What is the value of  $(3 \odot 5) \odot (2 \odot 4)$

174

$$(3 \odot 5) \odot (2 \odot 4) = 22 \odot 14 = 14^2 - 22 = 174$$

6. How many integers from 1 to 100 contain 5 as one of its digits but not a 3?

17

7. A train leaves Ithaca at 6:00am heading north at 40 miles per hour. At 7:00am, another train leaves Ithaca heading north on a parallel track at 50 miles per hour. At what time are both trains the same distance away from Ithaca?

11am

From 6am to 7am, the first train has traveled 40 miles. The difference in speeds between the trains is 10 miles per hour, so the second train needs four hours to catch up.

8. Suppose that  $19m$  is a perfect square. What is the smallest value of  $m$  such that  $19+m$  is a perfect square.

342

Since  $19m$  is a perfect square,  $m = 19n^2$ . Then,  $19 + m = 19 + 19n^2 = 19(1 + n^2)$  is a perfect square if  $1 + n^2$  is a multiple of 19. The smallest value of  $n$  for this is  $n = \sqrt{18}$ , then  $m = 19 \cdot 18 = 342$ .

9. What is the sum of all the roots of  $(3x - 2)(x + 5) + (2x + 3)(3x - 2) = 0$ ?

-2

$(3x - 2)(x + 5) + (2x + 3)(3x - 2) = 0 \implies (3x - 2)(3x + 8) = 0$ . Roots are  $2/3$  and  $-8/3$ , and their sum is  $-2$ .

10. If  $12 \cos(x) = 5 \sin(x)$ , compute  $\sin(2x)$ . Express your answer as a quotient of two integers.

120/169

$12 \cos(x) = 5 \sin(x) \implies \tan(x) = 12/5$ . 5 and 12 are legs of a 5-12-13 right triangle. Then,  $\sin(2x) = 2 \sin(x) \cos(x) = 2(12/13)(5/13) = 120/169$

11. A box contains either a white ball or a black ball. A black ball is then added to the box. You draw a ball from the box and it is black. What is the probability that the remaining ball is also black?

2/3

12. A motorcycle dealer is trying to sell a used bike and offers it on sale for 10% off the original price. When no one buys it, he offers 25% off the sales price. Again, no one buys it, and so he offers another 15% off of the second sales price. The bike sells for \$1836. What was the original price of the bike?

\$3200

Let  $x$  be the original price. Then,  $.85(.75(.9x)) = 1836 \implies x = 1836/(.57375) = 3200$ .

13. What is the smallest number that is divisible by 1, 2, 3, 4, 5, 6, 7, 8, and 9?

2520

The smallest is  $9 \cdot 8 \cdot 7 \cdot 5 = 2520$

14. A line with slope 5 intersects a line with slope  $-2$  at the point  $(10, 20)$ . How far apart are the  $x$ -intercepts of these lines?

14

The line with slope 5 has equation  $y - 20 = 5(x - 10)$  and  $x$ -intercept of 6. The line with slope  $-2$  has equation  $y - 20 = -2(x - 10)$  and  $x$ -intercept of 20. The difference is 14.

15. I have \$4.65 in quarters and dimes. If the dimes were quarters and the quarters were dimes, then I would have \$1.20 more. How many quarters do I have?

11

Let  $q$  be the number of quarters and let  $d$  be the number of dimes. In the first case,  $.25q + .10d = 4.65$  and in the second case,  $.25d + .10q = 5.85$ . Solving the system of equations yields  $q = 11$  and  $d = 19$ .

16. For real numbers  $a$  and  $b$ , we have  $\frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}} = 2020$ . What is the value of  $\frac{a+b}{a-b}$ ?

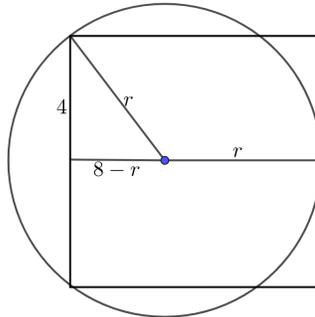
-2020

$$\frac{a+b}{a-b} = -\frac{a+b}{b-a} = -\frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}} = -2020$$

17. Consider the triangle  $\triangle ABC$  in the first quadrant formed by  $A = (0, 0)$  and the points  $B, C$  which are the intersections of the coordinate axes with the line  $x + 2y = 9$ . How many points  $(x, y)$  are on or inside  $\triangle ABC$  for which both coordinates are integers?

30

18. In the figure below, the square is tangent to the circle and intersects the circle in four other points. If the perimeter of the square is 32, what is the area enclosed by the circle?



25π

If  $r$  is the radius of the circle, we see that we can create a right triangle and solve for  $r$ :

$$(8 - r)^2 + 4^2 = r^2 \implies 64 - 16r + r^2 + 16 = r^2 \implies r = 5.$$

Thus, the area is  $5^2\pi$ .

19. What is the sum of the first 2022 odd numbers?

2022<sup>2</sup>

Observe the pattern:

$$\begin{aligned} \underbrace{1 + 3 + 5}_{3 \text{ odds}} &= 3^2 \\ \underbrace{1 + 3 + 5 + 7}_{4 \text{ odds}} &= 4^2 \\ \underbrace{1 + 3 + 5 + 7 + 9}_{5 \text{ odds}} &= 5^2 \end{aligned}$$

20. The first term in a sequence is 31. The next term in the sequence is the sum of the cube of each digit in the previous term; that is, the second term is  $3^3 + 1^3 = 28$ . Each of the remaining terms in my sequence is the sum of the cubes of the digits of the previous term. What is the 2022<sup>nd</sup> number in this sequence?

250

The terms of the sequence are 31, 28, 520, 133, 55, 250, 133, 55, 250, 133, 55, ... After the initial two two terms, the sequence repeats with period 3. The 2022<sup>nd</sup> term is 250.