2023 Math Exploration Day Team Competition - Answers

1. 27

2. 35.2

3. $\dfrac{143}{24}$

4. 1

5. 104°

6. 2023

7. −109

8. $\dfrac{11}{16}$

9. 9°

10. 6.75

11. 5

12. 20

13. 220

14. 0.5

15. $\dfrac{7}{5}$

16. Megan, by 8 yards

17. 77 and 90

18. $y = \dfrac{\ln(729)}{27}$

19. $3 \cdot \left(\dfrac{2024 \cdot 2023}{2} \right) = 6,141,828$

20. $12 + 8\sqrt{2} \approx 23.3$

2023 Math Exploration Day
Team Competition - Solutions

1. Twenty-five percent less than 60 is two-thirds more than what integer?

   $\boxed{27}$

   Twenty-five percent less than 60 equals $0.75 * 60 = 45$. So, 45 is two-thirds more than a number, call it x, meaning $\frac{5}{3}x = 45$. This means that $x = \frac{3}{5}(45) = 27$.

2. You drop a bouncy ball from 10-meter platform and each time it bounces back up exactly two-thirds of its previous height. What is the total vertical distance traveled by the ball when it reaches its highest point after the third bounce? Round to the nearest tenth of a meter.

   $\boxed{35.2}$

   The total vertical distance is given by

   $$
   10 + \frac{2}{3} \cdot 10 + \frac{2}{3} \cdot 10 + \frac{2}{3} \cdot \frac{2}{3} \cdot 10 + \frac{2}{3} \cdot \frac{2}{3} \cdot 10 + \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot 10
   $$
   $$
   = \quad 10 + \frac{4}{3} \cdot 10 + \frac{8}{9} \cdot 10 + \frac{8}{27} \cdot 10
   $$
   $$
   = \quad 10 \cdot \left(1 + \frac{4}{3} + \frac{8}{9} + \frac{8}{27}\right)
   $$
   $$
   = \quad 10 \cdot \left(\frac{27 + 36 + 24 + 8}{27}\right)
   $$
   $$
   = \quad 10 \cdot \left(\frac{95}{27}\right) \approx 35.2
   $$

3. Compute the reciprocal of the sum of the reciprocals of the two smallest 2-digit prime numbers. Express your answer in the form $\frac{a}{b}$, where a and b are whole numbers.

   $\boxed{\dfrac{143}{24}}$

   11 and 13 are the smallest 2-digit prime numbers, so the answer is

   $$
   \frac{1}{\frac{1}{11} + \frac{1}{13}} = \frac{1}{\frac{13+11}{11 \cdot 13}} = \frac{143}{24}
   $$

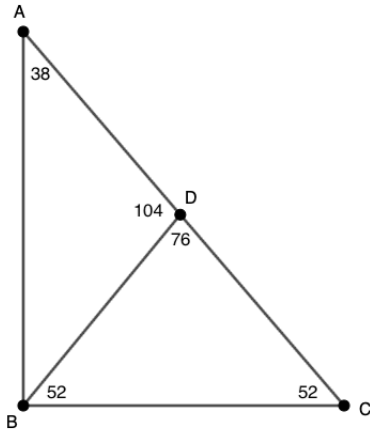4. Compute the sum of the roots of the equation $(2x + 4)(x - 2) + (x + 2)(x - 5) = 0$.

   $\boxed{1}$

   $$
   \begin{aligned}
   (2x + 4)(x - 2) + (x + 2)(x - 5) = 0 &\implies (x + 2)(2 \cdot (x - 2) + (x - 5)) = 0 \\
   &\implies (x + 2)(3x - 9) = 0
   \end{aligned}
   $$

   The roots are -2 and 3, so their sum is 1.

5. In right triangle ABC, the measure of angle BAC is $38°$. The length of segment BD equals the length of segment CD. What is the measure of the angle BDA?

$\boxed{104°}$

6. Given $f(x) = 1 - \dfrac{1}{x}$, compute $f(f(f(2023)))$. Provide your answer in simplified form.

$\boxed{2023}$

$$
\begin{aligned}
f(f(f(x))) &= f\left(f\left(1 - \frac{1}{x}\right)\right) \\
&= f\left(f\left(\frac{x-1}{x}\right)\right) \\
&= f\left(1 - \frac{x}{x-1}\right) \\
&= f\left(\frac{-1}{x-1}\right) \\
&= 1 - \frac{x-1}{-1} = x
\end{aligned}
$$

So, $f(f(f(2023))) = 2023$.

7. Define the operation $\star$ by $x \star y = x^2 + xy - y^2$. What is the value of $(2 \star 3) \star (4 \star 5)$

$\boxed{-109}$

$(2 \star 3) \star (4 \star 5) = (4 + 6 - 9) \star (16 + 20 - 25) = 1 \star 11 = 1 + 11 - 121 = -109$

8. A coin is flipped 4 times. What is the probability that at least 2 heads are obtained?

$\boxed{\frac{11}{16}}$

When flipping a coin 4 times, there are 16 possible outcomes. There is 1 way to have 4 heads, 4 ways to have 3 heads, and 6 ways to have 2 heads.

9. In a quadrilateral $ABCD$, we have:

   - the measure of $\angle B = 3 \times$ (the measure of $\angle A$),
   - the measure of $\angle C = 4 \times$ (the measure of $\angle B$), and
   - the measure of $\angle D = 2 \times$ (the measure of $\angle C$).

   What is the measure of $\angle A$?

   $\boxed{9°}$

   The sum of the interior angles of a quadrilateral is $360°$.

   $$
   \begin{aligned}
   \mathrm{m}\angle A + \mathrm{m}\angle B + \mathrm{m}\angle C + \mathrm{m}\angle D &= 360 \\
   \Longrightarrow \mathrm{m}\angle A + 3 \cdot \mathrm{m}\angle A + 12 \cdot \mathrm{m}\angle A + 24 \cdot \mathrm{m}\angle A &= 360 \\
   \Longrightarrow 40 \cdot \mathrm{m}\angle A &= 360 \\
   \Longrightarrow \mathrm{m}\angle A &= 9
   \end{aligned}
   $$

10. You hiked up a hill at rate of 1.5 miles per hour and came down at a rate of 4.5 miles per hour. It took you a total of 6 hours to make the hike up to the top and back down. How many miles was it to the top of the hill?

    $\boxed{6.75}$

    Let x be the distance (in miles) to the top of the hill. $x/1.5$ is the amount of time (in hours) it took you to hike up the hill and $x/4.5$ is the amount of time (in hours) it took you to hike down the hill. So, $x/1.5 + x/4.5 = 6$. Solve for x:

    $$x/1.5 + x/4.5 = 6 \Longrightarrow 4.5(x/1.5 + x/4.5) = 27 \Longrightarrow 3x + x = 27 \Longrightarrow x = \frac{27}{4} = 6.75$$

11. If you add the square of my dog's age to my cat's age, you get 14. If you add the square of my cat's age to my dog's age, you get 28. How old is my cat? Both ages are given in whole numbers of years.

    $\boxed{5}$

    Let c be my cat's age and let d be my dogs age. $d^2 + c = 14$ implies that $d = 1$ or $d = 2$ or $d = 3$. Then, $c^2 + d = 28$ implies that $c^2 = 28 - d$, meaning that $28 - d$ must be a perfect square. This happens only if $d = 3$ and $c = 5$.

12. Aaron's car averages 30 miles per gallon of gasoline and Osman's truck averages 15 miles per gallon of gasoline. If they both drive the same number of miles, what is their cars' combined rate of miles per gallon of gasoline?

    $\boxed{20}$

    If each drove 30 miles, then Aaron would use 1 gallon of gasoline and Osman would use 2 gallons of gasoline. Combined, they drive 60 miles and use 3 gallons of gasoline. So, their combined rate of miles per gallon is 60/3=20 miles per gallon.

13. Use the digits 1, 2, 3, 4, 6, 7, 8, and 9, to form 4 two-digit prime numbers, using each digit only once. What is the sum of the 4 prime numbers?

    $\boxed{220}$

    The 4 prime numbers must end in 1, 3, 7, and 9. So, 2, 4, 6, and 8 are used in the 10s place. The resulting some is $20 + 40 + 60 + 80 + 1 + 3 + 7 + 9 = 220$.

14. You are building a scale model of your town's water tower. It is 50 meters to the top of the town's tower. The top portion of the tower is a sphere that holds 100,000 liters of water. Your model's sphere holds 0.1 liters. How many meters tall should you make your tower?

$\boxed{0.5}$

The ratio of volumes is $\dfrac{100,000}{0.1} = 1,000,000$. The volume of a sphere is proportional to the cube of its radius. So, the ratio of radii of these water tower spheres is $\sqrt[3]{1,000,000} = 100$. So, the scale of the town to the model is 100:1. Thus, the model tower should be $50/100 = 0.5$ meters tall.

15. The sum of two positive numbers is 6 times their difference. What is the ratio of the larger number to the smaller number?

$\boxed{\dfrac{7}{5}}$

$$x + y = 6(x - y) \implies x + y = 6x - 6y \implies 7y = 5x \implies \frac{x}{y} = \frac{7}{5}.$$

16. The image shows a square field that has area of 40 acres. The line DE is a straight path and E is 110 yards away from B. In a race, Teresa runs directly from A to B, but Megan runs from D to E and then to B. Megan and Teresa begin the race at the same time. Each runs at their own constant speed and when Megan reaches E, Teresa is 30 yards ahead of her. Who wins the race, and by how much? (There are 4840 square yards in an acre.)

$\boxed{\text{Megan, by 8 yards}}$

40 acres = 193,600 square yards = 440^2 square yards. So, the field is 440 yards by 440 yards. When Megan reaches E, she has run 550 yards and Teresa has run 360 yards. Since there speeds are constant, the ratio of Megan's speed to Teresa's is 550/360. The winner of the race is the first person to arrive at B. Megan needs to run the final 110 yards (from E to B) if she is to win and Teresa needs to run another 80 yards if she is to win. Since the ratio of their speeds is constant, we can determine how far Teresa will have run (call this x) in the time it takes Megan to finish:

$$\frac{550}{360} = \frac{110}{x} \implies 550x = 360 \cdot 110 \implies x = \frac{39600}{550} = 72.$$

So, Megan wins the race by 8 yards since Teresa only runs 72 of her remaining 80 yards when Megan has already finished.
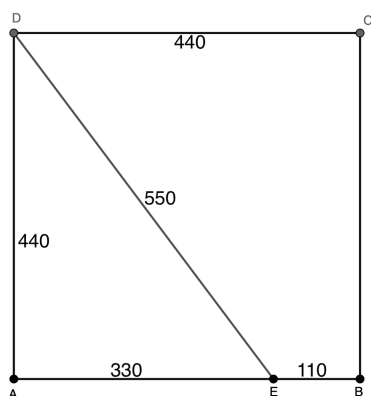
17. Find two positive integers whose product is 6930 and which have the smallest possible difference.

$\boxed{77 \text{ and } 90}$

The prime factorization of 6930 is $6930 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. We use this factorization to identify two factors which multiply to 6930 and are the closest together. They are $2 \cdot 3 \cdot 3 \cdot 5 = 90$ and $7 \cdot 11 = 77$.

18. Solve for y: $\ln(\ln(729)) - \ln y = \ln(27)$

$\boxed{y = \dfrac{\ln(729)}{27}}$

$$\ln(\ln(729)) - \ln y = \ln(27) \implies \ln\left(\frac{\ln(729)}{y}\right) = \ln(27)$$
$$\implies \frac{\ln(729)}{y} = 27$$
$$\implies y = \frac{\ln(729)}{27}$$

19. Starting with 3, what is the sum of the first 2023 whole numbers that are 3 apart? That is, what is $3 + 6 + 9 + \cdots$, ending with the 2023rd multiple of 3?

$\boxed{6{,}141{,}828}$

$$3 + 6 + 9 + \cdots + 3 \cdot 2023 = 3 \cdot (1 + 2 + 3 + \cdots + 2023) = 3 \cdot \left(\frac{2023 \cdot 2024}{2}\right) = 6,141,828$$

20. A circle of radius 2 sits tangent (just touching) the x- and y-axes. The length of line segment $\overline{CD}$ equals the length of line segment $\overline{DB}$. What is the area of triangle ABC?

$\boxed{12 + 8\sqrt{2} \approx 23.3}$

The conditions given imply that the segment $\overline{AD}$ is perpendicular to $\overline{BC}$. The height of $\triangle ABC$ is the length of $\overline{AD}$, which equals $2 + 2\sqrt{2}$. Also note that the length of $\overline{BD}$ equals the length of $\overline{CD}$ equals the length of $\overline{AD}$. So, the length of the base of $\triangle ABC$ equals $2 \cdot (2 + 2\sqrt{2}) = 4 + 4\sqrt{2}$. So, the area of the triangle is

$$\frac{1}{2}(4 + 4\sqrt{2})(2 + 2\sqrt{2}) = 12 + 8\sqrt{2} \approx 23.3$$