2024 Math Exploration Day
Team Competition - Answers

1. -5
2. 49
3. 5.4 hours, or 5 hours 24 minutes
4. 19
5. 8π
6. 4047
7. 1072
8. $\frac{5}{16}$
9. \$80
10. $\frac{35}{12}$
11. 55 gallons
12. 3840
13. 74 cm
14. 10
15. 9797
16. 6
17. 10
18. 382
19. $\frac{2\sqrt{3}}{3} - 1 = \frac{2\sqrt{3} - 3}{3} \approx 0.1547$
20. 1

2024 IC Math Exploration Day
Team Competition - Solutions

1. There is one negative integer whose square is 30 more than itself; which one?

   $\boxed{-5}$

   Let x be the integer. Then we have $x + 30 = x^2 \Longrightarrow x^2 - x - 30 = 0 \Longrightarrow (x-6)(x+5) = 0$.

2. Solve for x: $\log(\log(x)) - \log(\log(7)) = \log(2)$.

   $\boxed{49}$

$$\begin{aligned}
\log(\log(x)) - \log(\log(7)) = \log(2) &\Longrightarrow \log\left(\frac{\log(x)}{\log(7)}\right) = \log(2) \\
&\Longrightarrow \frac{\log(x)}{\log(7)} = 2 \\
&\Longrightarrow \log(x) = 2\log(7) \\
&\Longrightarrow \log(x) = \log(7^2) = \log(49) \\
&\Longrightarrow x = 49.
\end{aligned}$$

3. If it takes 6 students 9 hours to decorate the Math Department, how long will it take 10 students to complete the task?

   $\boxed{\text{5.4 hours, or 5 hours 24 minutes}}$

   It takes $6 \times 9 = 54$ student-hours to decorate the department, so 10 students will complete the decorating in $54/10 = 5.4$ hours, or equivalently, 5 hours 24 minutes.

4. The mean and median of a list of 5 positive integers are both 7. What is the largest possible number in the list?

   $\boxed{19}$

   Since the mean is 7, the sum of the five numbers must be 35. The smallest possible number in the list is 1. If 1 is used twice, then the first three numbers are: 1, 1, 7. The smallest that the fourth number can be is 7. Thus, 1, 1, 7, 7 is the smallest first four numbers possible. In this case, the largest possible is 35-(1+1+7+7)=19.

5. What is the area of the smallest circle that can be circumscribed about a square of area 16?

   $\boxed{8\pi}$

   The square has side-length 4, so the diagonal of the square has length $4\sqrt{2}$. The radius of the smallest circumscribed circle has length half the square's diagonal length, or $2\sqrt{2}$. The area of the circle is

$A = \pi(2\sqrt{2})^2 = 8\pi.$

6. 1 is the first odd natural number, 3 is the second, and so on. What is the 2024^{th} odd natural number?

   $\boxed{4047}$

   The n^{th} odd natural number is given by $2n - 1$, so the 2024^{th} odd natural number is 4047.

7. Define the operation $\star$ by $x \star y = x^3 - y^3$. What is the value of $(1 \star -2) \star (1 \star 2)$?

   $\boxed{1072}$

$$(1 \star -2) \star (1 \star 2) = 9 \star -7 = 9^3 - (-7)^3 = 729 + 343 = 1072.$$

8. You play a best of seven series of Monopoly games with an evenly-matched friend. This means that the first one to win four games is the champion. What is the probability that all seven games are required?

   $\boxed{\dfrac{5}{16}}$

   The last game does not matter. There are $\binom{6}{3} = 20$ possibilities, and since all of the possibilities involve each team winning three times, the probability of having 7 games is $20 \cdot \left(\dfrac{1}{2}\right)^6 = \dfrac{5}{16}$.

9. Teresa and Megan buy tickets to a play. Teresa buys six tickets using a coupon that gives her a 25% discount on each ticket and Megan buys four tickets using a coupon that gives her a 15% discount on each ticket. If Teresa paid a total of \$88 more than Megan, what is the price of each ticket?

   $\boxed{\$80}$

   Let p be the ticket price. Then,

$$6 \cdot 0.75 \cdot p - 4 \cdot 0.85 \cdot p = 88 \Longrightarrow p \cdot (4.50 - 3.40) \cdot p = 88 \Longrightarrow p = 88/1.1 = 80.$$

10. If $\tan(x) = 3/4$, then what is the value of $\csc(x) + \sec(x)$? Express the answer in the form a/b.

    $\boxed{\dfrac{35}{12}}$

    Consider a right triangle with base leg of length 4 and vertical leg of length 3. Then the hypotenuse has length 5. Let x be the angle between the base leg and hypotenuse. Then, $\tan(x) = 3/4, \sin(x) = 3/5$, and $\cos(x) = 4/5$. So,

$$\csc(x) + \sec(x) = \frac{1}{\sin(x)} + \frac{1}{\cos(x)} = \frac{5}{3} + \frac{5}{4} = \frac{35}{12}.$$

11. Joash filled a 550 gallon dunk tank for the county fair. Pete filled a child's play pool, using a hose with one-half the fill rate of Joash's hose. If it took Pete one-fifth the time to fill the pool that it took

Joash to fill the tank, how much water does the play pool contain?

$\boxed{\text{55 gallons}}$

Since Pete's hose fills at one half the rate of Joash's and it took one-fifth the time, the play pool has one-tenth the volume, or 55 gallons.

12. Given two numbers a and b, their average is given by $\frac{a+b}{2}$ and their harmonic mean is given by $\frac{2}{\frac{1}{a}+\frac{1}{b}}$. If the average of two numbers is 64 and their harmonic mean is 60, then what is the product of those two numbers?

$\boxed{3840}$

$$60 = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b} = ab \cdot \frac{2}{a+b} = ab \cdot \frac{1}{64} \Longrightarrow ab = 60 \cdot 64 = 3840.$$

13. The following shape is made up of three congruent rectangles, each with perimeter 37 cm, touching along some sides. What is the perimeter of the shape?

$\boxed{\text{74 cm}}$

The composite shape has two long sides matched up and two short sides matched up, leaving four long sides and four short sides as its edges. So, the perimeter is twice the perimeter of a single rectangle, or 74 cm.

14. Our classroom has a large jar of candies of different colors. The probability of choosing each color is given in the following table:

color	red	orange	yellow	green	blue
probability	0.13	0.30	0.17	0.15	

If you were to pick 40 candies from the jar, how many blue candies would you expect to pick?

$\boxed{10}$

The given probabilities sum to 0.75, so the probability of choosing a blue candy is 0.25. If we select 40 candies, we expect one-quarter of them (10) to be blue.

15. What is the product of the largest two-digit prime number and the smallest three-digit prime number?

$\boxed{9797}$

The largest two-digit prime is 97 and the smallest three-digit prime is 101. Their product is 9797.

16. Consider the sequence $a_1 = 4, a_2 = 6$, and for $n > 2$, we have $a_n = \dfrac{a_{n-1}}{a_{n-2}}$. What is the value of a_{2024}?

$\boxed{6}$

Compute the first few terms to get a sense of the sequence values:

$$a_3 = \frac{6}{4},\ a_4 = \frac{1}{4},\ a_5 = \frac{1}{6},\ a_6 = \frac{4}{6},\ a_7 = 4,\ a_8 = 6.$$

The sequence repeats itself every 6 steps. So $a_{2024} = a_{6\cdot 337+2} = a_2 = 6$.

17. Consider the curve given by the equation $36x^2 + 16y^2 = 576$. Let $x_{\max}$ be the largest x-coordinate of a point on the curve and let $y_{\max}$ be the largest y-coordinate of a point on the curve. What is the value of $x_{\max} + y_{\max}$?

$\boxed{10}$

The curve is an ellipse, so we can find $x_{\max}$ by setting $y = 0$ and solving for x: $36x^2 = 576 \implies x = \pm 4$. So, $x_{\max} = 4$. Analogously, $y_{\max} = 6$. Then, $x_{\max} + y_{\max} = 10$.

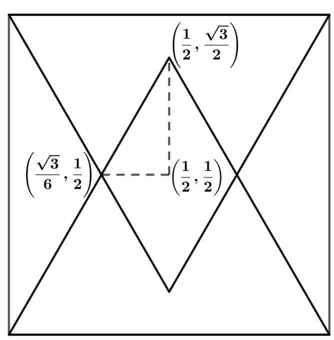
18. If $S_1 = 20 + 22 + 24 + \cdots + 400$ and $S_2 = 22 + 24 + 26 + \cdots + 402$, then what is the value of $S_2 - S_1$?

$\boxed{382}$

$S_2 - S_1 = 402 - 20 = 382$.

19. Two equilateral triangles have bases that are the opposite edges of a unit square, as in the image below. What is the area of overlap of the two triangles?

$\boxed{\dfrac{2\sqrt{3}}{3} - 1 = \dfrac{2\sqrt{3} - 3}{3} \approx 0.1547}$



Placing the lower left corner of the square at $(0, 0)$ allows us to identify coordinates of key points. By symmetry, the area of overlap is equal to four times the area of the triangle defined by the dotted lines. So, the area of the overlap is

$$4 \cdot \frac{1}{2} \cdot \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \frac{1}{6}(3 - \sqrt{3})(\sqrt{3} - 1) = \frac{1}{6}(4\sqrt{3} - 6) = \frac{2\sqrt{3}}{3} - 1$$

20. Given a positive integer n, we define $s(n)$ to be the sum of the digits of n. For how many values of n does $n + s(n) = 2024$?

$\boxed{1}$

Clearly, $n < 2024$. For $n < 2024$, the maximum value of $s(n)$ occurs when $n = 1999$; in this case, $s(1999) = 28$. So, $n > 2024-28 = 1996$. For $1996 < n < 2024$, a quick check shows that $n+s(n) = 2024$ only when $n = 2020$.