2025 IC Math Exploration Day
Team Competition - Answers

1. $\dfrac{9}{20}$
2. 7
3. $\dfrac{12}{5}$
4. 930
5. -1
6. 2025
7. 10
8. $\dfrac{1}{5} = 0.2$
9. 85
10. 6
11. 5
12. $\frac{1}{3}$
13. $5:7$, or $\dfrac{5}{7}$
14. 45
15. 15
16. $\dfrac{89}{55}$
17. $1+i$
18. 30
19. 1331
20. 3751

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Team Competition - Solutions

1. Compute $\dfrac{2+4+6+8}{1+3+5+7} - \dfrac{1+3+5+7}{2+4+6+8}$, expressing the answer in the form $\dfrac{a}{b}$ written in lowest terms.

   $\boxed{\dfrac{9}{20}}$

$$\frac{2+4+6+8}{1+3+5+7} - \frac{1+3+5+7}{2+4+6+8} = \frac{20}{16} - \frac{16}{20} = \frac{5}{4} - \frac{4}{5} = \frac{25}{20} - \frac{16}{20} = \frac{9}{20}.$$

2. The cube of a positive integer is 49 times that integer. What is that integer?

   $\boxed{7}$

   Let x be the unknown positive integer. Then, we have

$$x^3 = 49x \Longrightarrow x^3 - 49x = 0 \Longrightarrow x(x+7)(x-7) = 0 \Longrightarrow x = 0, -7, \text{ or } 7.$$

3. If $\sec(A) = \dfrac{13}{5}$, then what is the value of $\left|\tan(A)\right|$? Express your answer in the form $\dfrac{a}{b}$.

   $\boxed{\dfrac{12}{5}}$

   Since we want a non-negative answer, we may assume that we are working with a right-triangle situated in the first quadrant. $\sec(A) = \dfrac{13}{5}$ implies that the side adjacent to A has length 5 and the hypotenuse has length 13. By the Pythagorean Theorem, the side opposite A has length 12. So, $\left|\tan(A)\right| = \dfrac{12}{5}$.

4. Define the binary operation ☺ by $x☺y = x^2 \cdot y + y^2 \cdot x$. Compute $1☺(2☺3)$.

   $\boxed{930}$

$$1☺(2☺3) = 1☺(2^2 \cdot 3 + 3^2 \cdot 2) = 1☺30 = 1^2 \cdot 30 + 30^2 \cdot 1 = 930.$$

5. Find the sum of all solutions of $\left|x+1\right| + \left||x| - 2\right| = 5$.

   $\boxed{-1}$

   The solutions to this equation can be found by graphing or by inspecting the cases $x < 0$ and $x > 0$. The solutions are $x = -4$ and $x = 3$, so the sum is -1.

6. The geometric mean of three numbers (say, a, b, c) is $\sqrt[3]{a \cdot b \cdot c}$. If the geometric mean of $3, 15$, and c is 45, then what is the value of c?

   $\boxed{2025}$

$$\sqrt[3]{3 \cdot 15 \cdot c} = 45 \Longrightarrow 45c = 45^3 \Longrightarrow c = 45^2 = 2025.$$

7. The vertices of triangle ABC lie on the parabola $y = 4x^2$ with A at $(0, 0)$ and the side $\overline{BC}$ perpendicular to the y-axis. If the area of the triangle is 500, what is the length of $\overline{BC}$?

$\boxed{10}$

Let h be the height of the triangle and let b be the length of the base, which is the side $\overline{BC}$. Let $(a, 4a^2)$ be the coordinates of right-end of $\overline{BC}$. Then, we have

$$\frac{1}{2} \cdot b \cdot h = 500 \Longrightarrow \frac{1}{2} \cdot 2a \cdot 4a^2 = 500 \Longrightarrow 4a^3 = 500 \Longrightarrow a^3 = 125 \Longrightarrow a = 5.$$

This means that the length of $\overline{BC}$ is 10.

8. A bag has 6 M&Ms remaining: 2 green, 2 yellow, and 2 red. Armin takes two at random, then Beth takes two at random, and Carlo is left with the final two. What is the probability that Carlo gets two M&Ms of the same color?

$\boxed{\frac{1}{5} = 0.2}$

This probability is the same as if Carlo went first and picked two of the same color. That probability is 1/5.

An alternative reason follows.

9. The average grade in my morning Algebra I class of 30 students is 81. The average grade in my afternoon Algebra I class of 24 students is 90. What is the average grade of all students in these morning and afternoon classes?

$\boxed{85}$

$$\text{The average for both classes} = \frac{30 \cdot 81 + 24 \cdot 90}{30 + 24} = 85.$$

10. In their latest game, the IC women's basketball team was successful on 60% of their 2-point shots and 30% of their 3-point shots. This yielded 66 points. If they attempted twice as many 2-point shots as 3-point shots, how many successful 3-point shots did they make?

$\boxed{6}$

Let x be the number of 2-point shots attempted and let y be the number of 3-point shots attempted. We are given $x = 2y$ and that $0.6 \cdot 2 \cdot x + 0.3 \cdot 3 \cdot y = 66$. Substituting $x = 2y$, we have

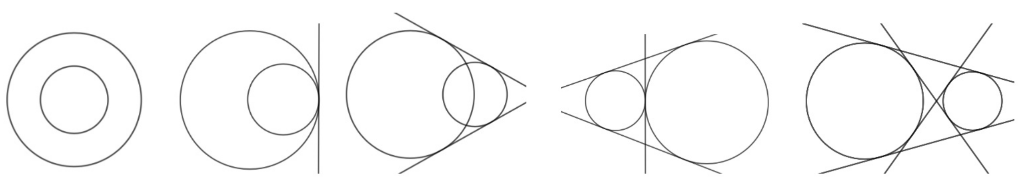
$$1.2 \cdot 2y + 0.9y = 66 \Longrightarrow 3.3y = 66 \Longrightarrow y = 20.$$

So, they attempted 20 3-point shots and make 30% of them, which is 6.

11. Each student in a geometry class are asked to draw a circle of radius 2 and a circle of radius 4 on a page in their notebook. Then, they are asked to draw all possible lines that are simultaneously tangent to both circles and record the number of lines they see. How many different correct answers are possible?

$\boxed{5}$

The possible configurations are shown below.



12. Gabbie and Michael play a game in which they each flip a fair coin. Each one flips their coin until it lands heads up and then they stop. What is the probability that both of them flip their coins the same number of times?

$\boxed{\frac{1}{3}}$

Let P be the probability that both flip their coins the same number of times. There is $1/4$ chance that they both flips heads on the first flip. If they don't both flip heads on the first turn, the only way for them to flip the same number of times is if they both flipped tails one the first flip, which occurs with probability $1/4$. At this point, it is the same as being back to the start, and these are the only ways to have the same number of flips. This means that $\frac{1}{4} + \frac{1}{4}P = P$. So, $P = \frac{1}{3}$.

A second way to compute this is with an infinite sum. We've already seen that there is a $1/4$ probability of having the same number of flips after one flip. There is a $(1/4)^2$ probability of getting exactly two flips, $(1/4)^3$ probability of exactly three flips, and so on. Thus, the probability is

$$P = \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + + \left(\frac{1}{4}\right)^4 + \cdots = \frac{1}{3}.$$

13. The sum of two positive integers is six times their difference. What is the ratio of the smaller number to the larger?

$\boxed{5 : 7, \text{ or } \frac{5}{7}}$

Let $x > y$ be the two positive integers.

$$x + y = 6(x - y) \Longrightarrow 5x = 7y \Longrightarrow \frac{y}{x} = \frac{5}{7}.$$

14. The interval of solutions of the inequality $m \leq 3x + 5 \leq n$ has length 15. What is $n - m$?

$\boxed{45}$

$$m \leq 3x + 5 \leq n \Longrightarrow \frac{m-5}{3} \leq x \leq \frac{n-5}{3}.$$

The interval of solutions has length 15, so

$$15 = \frac{n-5}{3} - \frac{m-5}{3} = \frac{n-m}{3} \Longrightarrow n - m = 3 \cdot 15 = 45.$$

15. How many positive perfect squares are there that are less than or equal to 2025 and divisible by 3?

$\boxed{15}$

Note that $2025 = 45^2$. So, there are 45 perfect squares less than or equal to 2025, and one-third of those will be divisible by 3.

16. Consider the sequence $a_1 = 1$ and for $n > 1$, we have $a_n = 1 + \frac{1}{a_{n-1}}$. What is the value of a_{10}? Express your answer in the form $\frac{a}{b}$.

$\boxed{\frac{89}{55}}$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 + 1 = 2 \\ a_3 &= 1 + \frac{1}{2} = \frac{3}{2} \\ a_4 &= 1 + \frac{2}{3} = \frac{5}{3} \\ a_5 &= 1 + \frac{3}{5} = \frac{8}{5} \end{aligned}$$

Notice that these are the ratios of consecutive Fibonacci numbers. So, $a_{10} = \frac{89}{55}$.

17. The imaginary number, i, is defined by $i^2 = -1$. What is the value of the following sum:

$$\sum_{k=0}^{2025} i^k = 1 + i + i^2 + i^3 + i^4 + i^5 + \cdots + i^{2024} + i^{2025}?$$

$\boxed{1 + i}$

$$\begin{aligned} \sum_{k=0}^{2025} i^k &= 1 + i + i^2 + i^3 + i^4 + i^5 + \cdots + i^{2024} + i^{2025} \\ &= (1 + i + i^2 + i^3) + (i^4 + i^5 + i^6 + i^7) + \cdots + (i^{2020} + i^{2021} + i^{2023} + i^{2024}) + i^{2024} + i^{2025} \\ &= (1 + i - 1 - i) + (1 + i - 1 - i) + \cdots + (1 + i - 1 - i) + 1 + i \\ &= 0 + 0 + \cdots + 0 + 1 + i \\ &= 1 + i \end{aligned}$$

18. What is the value of x for $\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} = 1$?

$\boxed{30}$

We can use the change of base formula for logarithms. For example, $\log_2 x = \frac{\ln x}{\ln 2}$.

$$\begin{aligned} \frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} = 1 \quad &\Longrightarrow \quad \frac{\ln 2}{\ln x} + \frac{\ln 3}{\ln x} + \frac{\ln 5}{\ln x} = 1 \\ &\Longrightarrow \quad \ln 2 + \ln 3 + \ln 5 = \ln x \\ &\Longrightarrow \quad \ln 30 = \ln x \\ &\Longrightarrow \quad x = 30. \end{aligned}$$

19. The numbers 3, 4, 5, 6, 7, and 8 are placed one each on the faces of a six-sided die. For each of the eight vertices of the die, compute the product of the numbers assigned to the three faces that contain that vertex. What is the largest possible sum of the eight products?

$\boxed{1331}$

Label the six faces using a, b, c, d, e, and f, with f opposite a and b through d in order around the other four sides. The sum of the products is represented by

$$abc + acd + ade + abe + fbc + fcd + fde + fbe = (a + f)(b + d)(c + e).$$

The sum is maximized when $a + f = 3 + 8 = 11, b + d = 4 + 7 = 11$, and $c + e = 5 + 6 = 11$. Then, $11^3 = 1331$.

20. $\sigma(n) =$ the sum of all positive factors of n. For example, $\sigma(12) = 28$ because the positive factors of 12 are 1, 2, 3, 4, 6, and 12. Compute $\sigma(2025)$.

$\boxed{3751}$

The positive factors of 2025 are 1, 3, 5, 9, 15, 25, 27, 45, 75, 81, 135, 225, 405, 675, and 2025. So, $\sigma(2025) = 3751$.