Digital Roots: Insights from Ancient Indian Multiplication

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Math Explorations Day
Teachers Talk

Activity 1

Follow the instructions below:

1. Choose any number greater than two digits.
2. Add up its digits.
3. Subtract the sum from the original number.
4. Divide the difference by nine.
5. Concentrate on your remainder.

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Discussion of Activity 1

(a) Consider a two-digit number represented by ab. This number is the short form of

$$a \cdot 10^1 + b \cdot 10^0 = 10a + b.$$

When we add up its digits, we obtain $a + b$. Subtracting the sum from the original number, we have

$$(10a + b) - (a + b) = 9a.$$

Therefore, the answer is a multiple of 9.

(b) Consider a two-digit number represented by abc. Express abc using the powers 10

$$a \cdot 100 + b \cdot 10 + c = 100a + 10b + c$$

When we add up its digits, we obtain $a + b + c$. Subtracting the sum from the original number, we have

$$(100a + 10b + c) - (a + b + c) = 99a + 9b = 9(9a + b).$$

Therefore, the answer is a multiple of 9.
This will be correct for any digit of the number.

Activity 2

1. Secretly choose one of the four digits numbers:

   $$3141 \quad \text{or} \quad 2718 \quad \text{or} \quad 2358 \quad \text{or} \quad 9999$$

   These numbers, respectively, the first four digits of π, the first four digits of e, consecutive Fibonacci numbers, and the largest four-digit number.
2. Multiply your number by any three-digit number.
3. Circle one of the digits of their number but not circle a 0.
4. Recite all of the uncircled digits in any order.
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Discussion of Activity 2

Notice that all four numbers they could start with are multiples of 9. Since you begin with a multiple of 9 and multiply it by a whole number, the answer will still be a multiple of 9. Therefore, its digits must add up to a multiple of 9.
As the numbers are called out to you, simply add them up. The missing digit is the number you need to add to your total to reach a multiple of 9.
For instance, suppose the spectator calls out the digits

$$5, 0, 2, 2, 6, 1 \implies \text{Sum} = 16.$$

Thus they must have left out the number 2 to reach the nearest multiple of 9, which is 18.
If the numbers called out are

$$1, 1, 2, 3, 5, 8 \implies \text{Sum} = 20,$$

then the missing digit must be 7 to reach 27.
Suppose the numbers called out add up to 18. What did they leave out? Since we instructed them not to focus on 0, the missing digit must be 9.

What is so magical about number 9?

Let us look at the multiples of 9:

$$9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, 135, 144, \ldots$$

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135: Digit sum $1 + 3 + 5 = 9$

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Thus, in general, any number with a digit sum that is a multiple of 9 must itself be a multiple of 9 and vice versa.

Digital Root and Casting of Nines

We give an example with the number 4783. We expressed the number using the powers of 10:

$$\begin{aligned}
4783 &= 4\cdot 1000 + 7\cdot 100 + 8\cdot 10 + 3 \\
&= 4\cdot(999+1) + 7\cdot(99+1) + 8\cdot(9+1) + 3 \\
&= 4(999) + 7(99) + 8(9) + 4 + 7 + 8 + 3 \\
&= 9[4(111) + 7(11) + 8(1)] + 22 \\
&= 9a + 2(9+1) + 2 \\
&= 9(a+2) + 4 \\
&= 9b + 4
\end{aligned}$$

Thus the remainder is 4 when we divide 4783 by 9. If we keep adding the digits 9 until we get a one digit number we also get the same answer 4. We represent this calculation as

$$4783 \to 22 \to 4.$$

The number 4 is called the digital root of 4783. The process of adding the digits of your number and repeating that process until you are reduced to a one-digit number is called **asting out nines.**

Fundamental Principal of Digital Roots

If n has a digital root of 9, then n is a multiple of 9.
Otherwise, the digital root is the remainder obtained when n is divided by 9.
Aletrnatively, we can expressed this fact:

If n has digital root r, then

$$n = 9x + r$$

for some integer x.

Activity 3

Follow the instructions below:

1. Choose any three-digit number where the first and last digits are different (e.g., 842).
2. Reverse the digits of the number (e.g., 248).
3. Subtract the smaller number from the larger number.
4. Reverse the digits of the difference.
5. Add the reversed difference to the original difference.
6. Concentrate on your sum.

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Discussion of Activity 3

Let abc be our selection. Without loss of generality, we assume that $a > c$. We express the number using powers of 10:

$$abc = a \cdot 100 + b \cdot 10 + c = 100a + 10b + c.$$

Let cba be the rearrangement. Expressing as powers of 10 yields

$$cba = c \cdot 100 + b \cdot 10 + a = 100c + 10b + a.$$

Based on the assumption abc is greater than cba and subtracting the smaller number cba from the larger number abc gives rise to

$$\begin{aligned} abc - cba &= (100a + 10b + c) - (100c + 10b + a) \\ &= 99a - 99c = 99(a - c). \end{aligned}$$

Since $a \neq c$, $a - c$ is a nonzero digit from 1 to 9. Therefore, the only possibilities are

$$99, 198, 297, 396, 495, 594, 693, 792, 891, 990.$$

In each case, the result is

$$990 + 99 = 1089, \quad 198 + 891 = 1089, \quad 297 + 792 = 1089, \quad 396 + 693 = 1089, \quad 495 + 594 = 1089.$$

Digital root of the sum of the number from 1 to 9

We have

$$1+2+3+\cdots+9=\frac{9\cdot 10}{2}=9\cdot 5=45.$$

Thus the digital root of the sum is $4+5=9$.

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In Vedic numerology, 45 represents balance and harmony

The Vedic Square

An Indian multiplication table called Vedic square is formed similar to multiplication table, instead placing the product of the number, their digital roots are placed. As a result we obtain the following table.

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	1	3	5	7	9
3	3	6	9	3	6	9	3	6	9
4	4	8	3	7	2	6	1	5	9
5	5	1	6	2	7	3	8	4	9
6	6	3	9	6	3	9	6	3	9
7	7	5	3	1	8	6	4	2	9
8	8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9	9

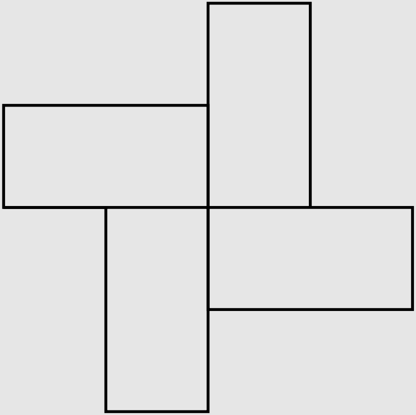
Properties of the Vedic Square

This table has a lot of properties. We will investigate some of its properties. We give the table without the top row and the first column.

1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

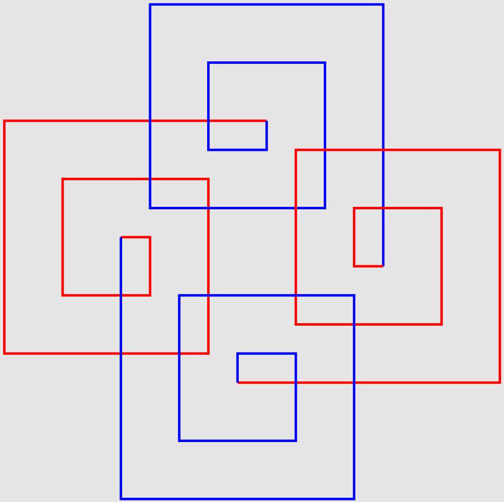
The Third Row of the Vedic Square

Start anywhere from middle of a grid paper and move 3 units in one direction (up, down, right, or left), then move 6 unit right and continue with the sequence $\{3, 6, 9, 3, 6, 9, 3, 6, 9\}$ of the third row. You must continue until you reached the point you started. Eventually, we obtain the geometric figure



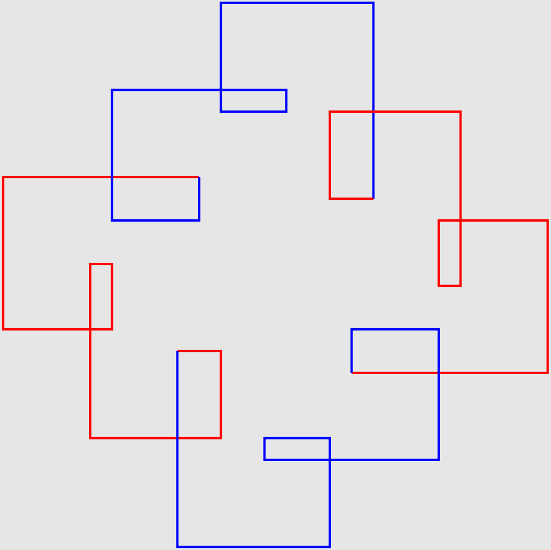
The First Row of the Vedic Square

This is similar to the previous idea, using the sequence $\{1,2,3,4,5,6,7,8,9\}$ of the first row, we obtain the geometric figure



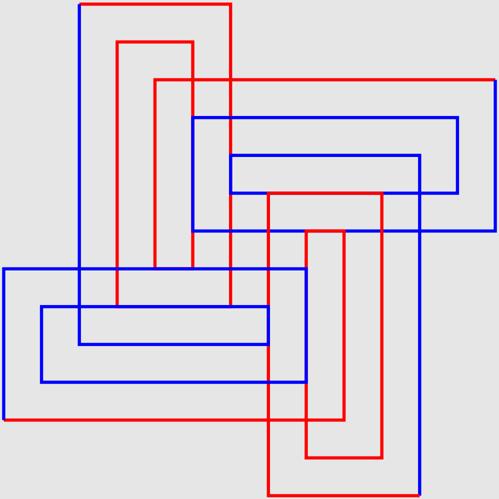
The Second Row of the Vedic Square

If we use the second row $\{2, 4, 6, 8, 1, 3, 5, 7, 9\}$, we obtain the figure



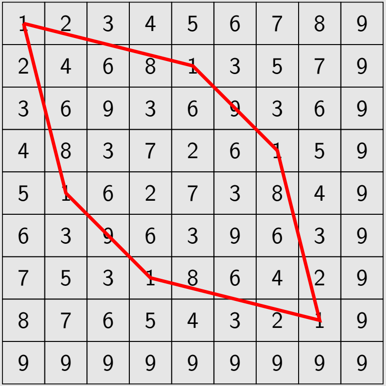
The Fourth Row of the Vedic Square

If we use the fourth row $\{4, 8, 3, 7, 2, 6, 1, 5, 9\}$, we obtain the figure

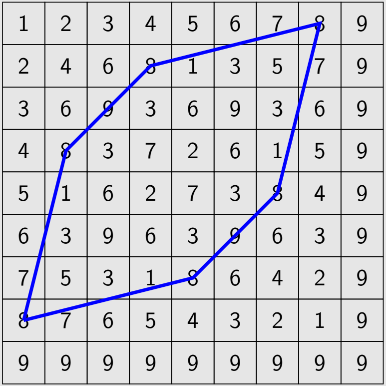


Designs Created by the number 1 and 8 from the Vedic Square

We consider numbers 1 and 8 and identify them in the box and connect them with straight lines as show in the following figures.



1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9



1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

Designs Created by the number 2 and 7 from the Vedic Square

We consider numbers 2 and 7 and identify them in the box and connect them with straight lines as show in the following figures.

1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

Designs Created by the number 3 and 6 from the Vedic Square

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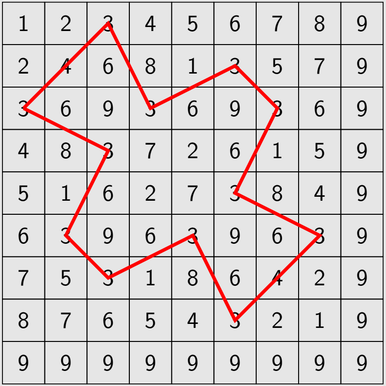
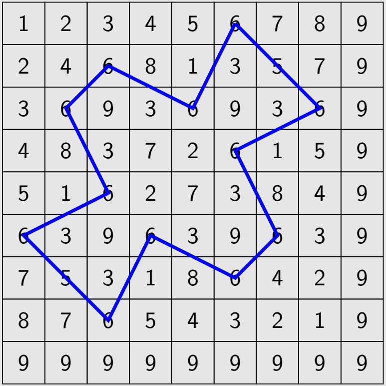
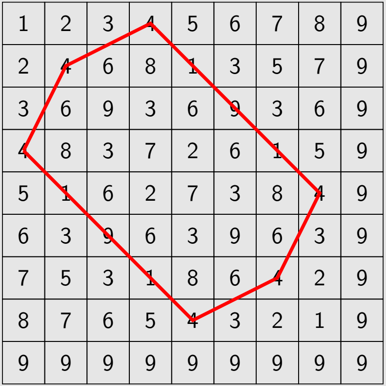


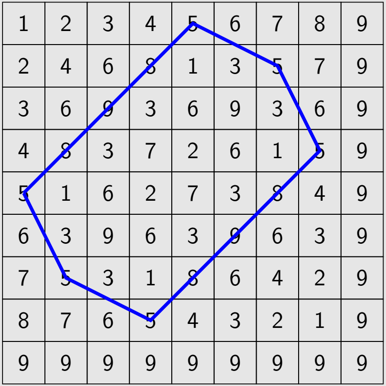


Designs Created by the number 4 and 5 from the Vedic Square

We consider numbers 4 and 5 and identify them in the box and connect them with straight lines as show in the following figures.



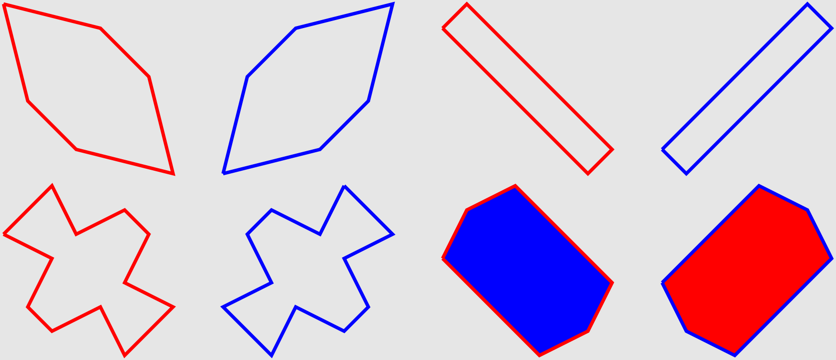
1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9



1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

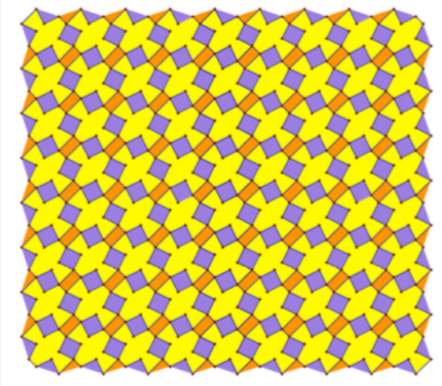
All Designs Created from the Vedic Square

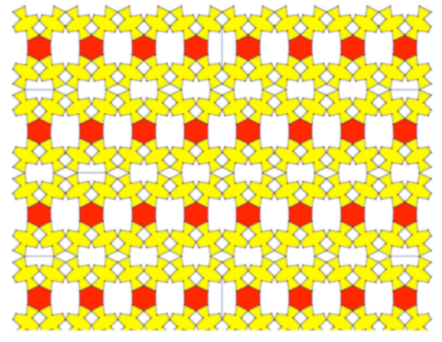
We separate the shapes from their squares and put them together. We obtain all the figures below.



Tessellations from 3 and 8

By considering some of these figures and using their repeating designs, it is possible to obtain tessellations. We give two samples.





Islamic Designs

The following are the actual Islamic designs.

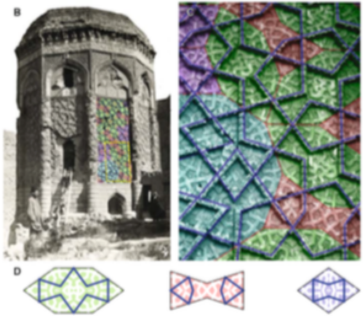


Fig. 2. (**A**) Periodic girih pattern from the Seljuk Mama Hatun Mausoleum in Tercan, Turkey (~1200 C.E.), where all lines are parallel to the sides of a regular pentagon, even though no decagon star is present; reconstruction overlaid at right with the hexagon and bowtie girih tiles of Fig. 1F. (**B**) Photograph by A. Sevruguin (~1870s) of the octagonal Gunbad-i Kabud tomb tower in Maragha, Iran (1197 C.E.), with the girih-tile reconstruction of one panel overlaid. (**C**) Close-up of the area marked by the dotted yellow rectangle in (B). (**D**) Hexagon, bowtie, and rhombus girih tiles with additional small-brick pattern reconstruction (indicated in white) that conforms not to the pentagonal geometry of the overall pattern, but to the internal two-fold rotational symmetry of the individual girih tiles.