INTERNAL RAYS OF THE MANDELBROT SET

HONORS THESIS
ITHACA COLLEGE

DEPARTMENT OF MATHEMATICS

BY
WALTER HANNAH

FACULTY MENTOR: DAVID BROWN

ITHACA, NY
APRIL 2006

ABSTRACT

This paper will investigate a method for seeing the internal structure of individual hyperbolic components of the Mandelbrot set by mapping the unit to disc to individual components. This will be used to explain how two components are connected to each other.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Background

1.1.1 Basics of Real Dynamics

In studying the dynamics of a function, we wish to understand the asymptotic behavior of points under iteration of the function. Suppose we have a set S and a function $f : S \to S$.

Definition 1. *If n is a positive integer, then the n-fold composition of f, $f^{\circ n}(x) = f(f(...(f(x)))$ is called the nth iterate of f.*

For any $x \in S$, we want to understand the sequence of points $\{x, f(x), f(f(x)), ...\}$. Does this sequence converge? If not, does it diverge? Does it contain any convergent subsequences? We call the set of numbers in this sequence the forward orbit of x under f.

Definition 2. *The forward orbit of x under f is the set of points $\{x, f(x), f(f(x)), ...\}$.*

We can see that if $f(x) = x$, then the forward orbit of x is just the set $\{x\}$. If $f^{\circ k}(x) = x$ for some integer k but $f^{\circ i}(x) \neq x$ for $i < k$, then the forward orbit of x is the set $\{x, f(x), ..., f^{\circ(k-1)}(x)\}$.

Definition 3. *The point x is a periodic point of prime period k if k is the smallest positive integer such that $f^{\circ k}(x) = x$. If $k = 1$ then x is a fixed point of f.*

So, if x is a periodic point of prime period k, then x is a fixed point of the function $f^{\circ k}$. If x is a periodic point of prime period k, then f has a k-cycle, containing the set of points $\{x, f(x), f^{\circ 2}(x), ..., f^{\circ(k-1)}(x)\}$. Choosing any element of this k-cycle and iterating f will produce the cycle.

If f is differentiable at a fixed point x, then the derivative of f at x provides some information about the orbits of points near x:

Definition 4. *If* $\left|f'(x)\right| < 1$ *then* x *is an attracting fixed point of* f*. That is, there is an open interval* U *around* x *such that* $f(U) \subset U$*. If* $f'(x) = 0$ *then* x *is a superattracting fixed point of* f*.*

Definition 5. *If* $\left|f'(x)\right| > 1$ *then* x *is a repelling fixed point of* f*.*

Definition 6. *If* $\left|f'(x)\right| = 1$ *then* x *is a non-hyperbolic, or indifferent, fixed point of* f*.*

We can think of repelling fixed points as “sources” and attracting fixed points as “sinks”; points near an attracting fixed point are drawn into the fixed point, while points near a repelling fixed point are pushed away from the fixed point. Note that cycles can also be sources and sinks: to determine if a k-cycle of f is attracting or repelling, we look at the value of $(f^{\circ k})'$ at any point in the cycle. Indifferent fixed points and cycles may be sources, sinks, or neither.

1.1.2 Basics of Complex Dynamics

Working in the complex plane, it is difficult to graph functions, since a complete picture would require 4-dimensional space. However, using the image of the Julia set we can observe the effects of attracting fixed points.

Definition 7. *A set* $\{f_n\}$ *of analytic functions defined on an open set* U *is a normal family on* U *if every sequence of the* f_n*'s contains a subsequence which converges uniformly on compact subsets of* U*. [1]*

As an example, we look at the quadratic function $Q(z) = z^2$. The set $\{Q^{\circ n}(z)\}$ is normal on the open unit disk $\{z_0 : |z_0| < 1\}$, and also on the set $\{z_0 : |z_0| > 1\}$, but not on any set containing some z_0 with $|z_0| = 1$.

Definition 8. *The Fatou set of* f *(also called the normal set or stable set) is the subset of* $\mathbb{C}$ *on which* $\{f^{\circ n}\}$ *is normal.*

The orbits of points in the Fatou set are similar to those of nearby points. If f is a polynomial, all points in the Fatou set will either be attracted to a fixed point (or cycle) or will diverge to infinity.

Definition 9. *The filled Julia set of a polynomial* f*, denoted* K_f*, is the set of all points with bounded orbits under* f*.*

Notice that K_f is not necessarily a subset of the Fatou set. In fact, the boundary of K_f is not in the Fatou set.

Definition 10. *The Julia set of* f*, denoted* J_f*, is the boundary of the filled Julia set, and is also the complement of the Fatou set.*

The Julia set J_f has the following properties:

1. If f is a polynomial, then J_f is the boundary of K_f.
2. The Julia set is completely invariant. That is, $f(J_f) \subseteq J_f$ and $f^{-1}(J_f) \subseteq J_f$.
3. If f is rational, then J_f is either nowhere dense or is the entire Riemann sphere $\hat{\mathbb{C}}$.

4. If $z_0 \in J_f$ and U is some open set containing z_0 then

$$\bigcup_{n=1}^{\infty} f^n(U)$$

is the entire complex plane, with the exception of at most one point.

We can see from the above properties that the Julia set is chaotic. That is, the orbit of $z \in J_f$ is very different from the orbits of nearby points.

From calculus we learn the importance of critical points, or the points at which the derivative of a function is equal to 0. Similarly, in complex dynamics critical points and their images will tell us a lot about the behavior of a function.

Definition 11. *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. Then z is a critical value of f if there exists some $z_0 \in \mathbb{C}$ such that $f(z_0) = z$ and $f'(z_0) = 0$. That is, critical values are the images of critical points.*

Critical values play an important role in the study of dynamics. If f has an attracting fixed point or an attracting cycle, it must attract a critical value of f.

1.2 The Mandelbrot set

When studying Julia sets, we notice that some are connected sets while others are disconnected; see Figure 1.1. We begin with this topological definition.

Definition 12. *A set S is disconnected if there exist non-empty sets A and B such that $A \bigcup B = S$, $\bar{A} \bigcap B = \emptyset$ and $A \bigcap \bar{B} = \emptyset$. If S is not disconnected then it is connected.*

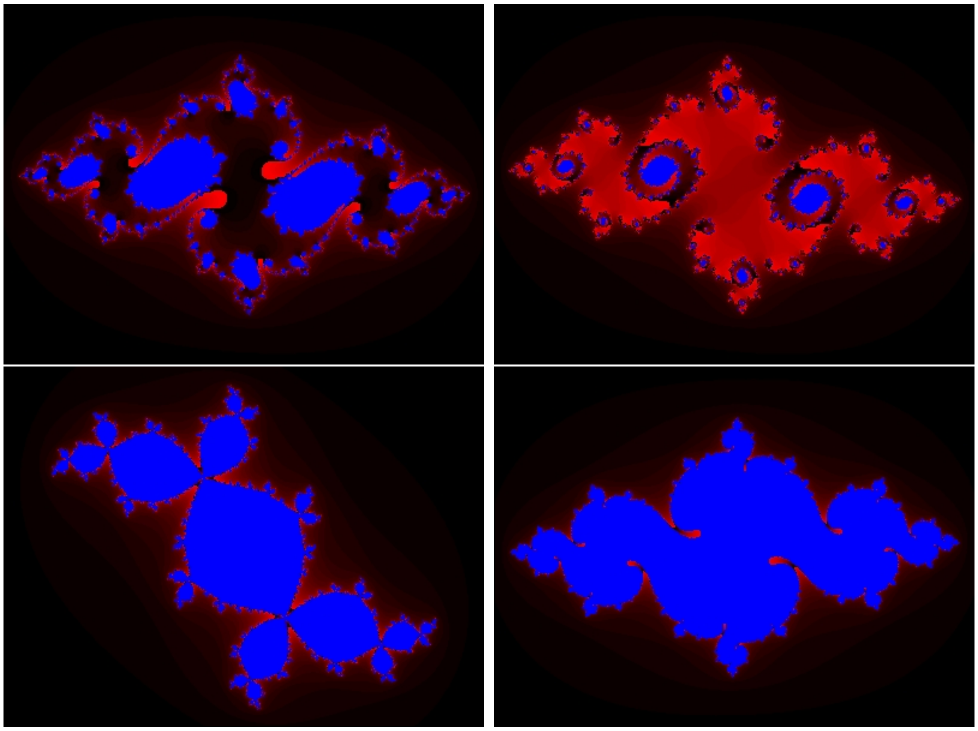


Figure 1.1: Examples of disconnected and connected Julia sets. Top two are disconnected and bottom two are connected.

Since a Julia set is either connected or completely disconnected, the choice of parameter c is playing a distinguished role. Those parameters that lead to connected Julia sets define the Mandelbrot set.

Definition 13. *Let* $Q_c(z) = z^2 + c$ *where* $c \in \mathbb{C}$*, then the Mandelbrot set is given by*

$$\mathcal{M} = \{c \mid K_{Q_c} \textit{is connected}\}$$

The following result is credited to Fatou.

Theorem 1. *Let* $f : \mathbb{C} \to \mathbb{C}$*. If all critical points of* f *are bounded, then* K_f *is connected.*

Drawing on our previous example where $Q(z) = z^2$, notice that 0 is our only critical point and when iterated the forward orbit of 0 is most definitely bounded. If we now consider all quadratic polynomials of the form $Q(z) = z^2 + c$, where $c \in \mathbb{C}$, then different values of c will give us different forward orbits of the critical point.

So by the previous result we can redefine $\mathcal{M}$.

Corollary 2. *The Mandelbrot set is the set of all $c \in \mathbb{C}$ such that the forward orbit of 0 is bounded.*

Proof. Since 0 is the only critical point of $Q_c(z)$ and its orbit is bounded, we have that $c = Q_c(0)$ lies in the Mandelbrot set. □

This definition gives us a simpler method of determining whether a point is in the Mandelbrot set, but first we need to know what constitutes a bounded orbit.

Theorem 3. *If $|z| > 2$ and $|c| \geq |z|$, then $Q_c^{\circ n}(z) \to \infty$ as $n \to \infty$.*

Proof. Let $|z| > 2$ and set $z_n = Q_c^{\circ n}(z)$. If $|c| \geq |z|$, then

$$\begin{aligned}
|z_1| &= |z^2 + c| \\
&\geq |z|^2 - |c| \quad \text{by the triangle inequality} \\
&\geq |z|^2 - |z| \\
&= (|z| - 1)|z| \\
&= \lambda|z| \quad \text{where } \lambda = |z| - 1 > 1 \\
&> |z| > 2.
\end{aligned}$$

Thus, $|z_1| > |z| > 2$, and by induction we find that

$$|z_n| > \lambda^n |z|.$$

So, as $n \to \infty$, we have $z_n \to \infty$. $\square$

Corollary 4. $\mathcal{M} \subseteq \{c : |c| \leq 2\}$

Proof. If $|c| > 2$, then $Q_c^{\circ n}(c) \to \infty$ and hence $c \notin \mathcal{M}$. $\square$

Now we have a way to estimate whether $Q_c(z)$ is bounded for a given c by checking if the first 50 to 100 iterations of the critical point $z = 0$ stay within a circle of radius 2, thereby determining whether a given c is in the Mandelbrot set. By using a coloring scheme based on whether orbits escape this circle we can build a picture of $\mathcal{M}$; see Figure 1.2.

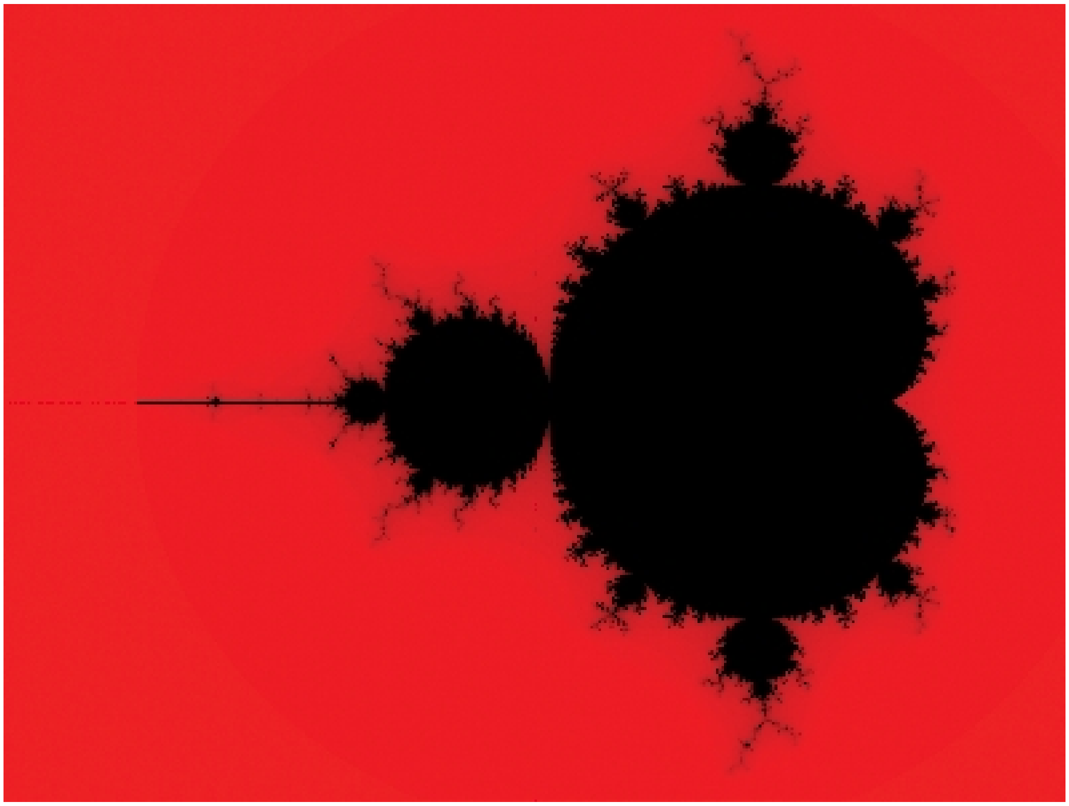


Figure 1.2: The Mandelbrot set.

1.3 Mapping the Unit Disc to M

1.3.1 $\mu(\lambda) = c$

Definition 14. *A polynomial is hyperbolic if each critical point is attracted by an attracting periodic cycle.*

Definition 15. *A component, W, of the interior o μ is called a hyperbolic component if Q_c is hyperbolic for all $c \in W$. We say a hyperbolic component W_N has a period N if Q_c has an attracting cycle of prime period N for all $c \in W_N$. These hyperbolic components are often called bulbs.*

In order to understand the Mandelbrot set more we can look at the internal geometry of individual bulbs. For a given bulb W we can define the complex mapping $\mu : \mathbb{D} \mapsto W$ by $\mu(\lambda) = c$, where $c \in W$ is the solution to

$$\begin{aligned} Q_c^n(z) &= z \\ (Q_c^n)'(z) &= \lambda \end{aligned}$$

Since this task would be near impossible by hand, the assistance of a computer becomes necessary even though we will see that the calculations quickly become difficult for the computer as well. Notice that $\mu(\lambda)$ maps complex numbers to complex numbers. Since a computer can only use real numbers we can parameterize a polar grid to visualize the map. To parameterize a circle we want all $\lambda = x + \imath y$ with a certain radius which can be found by

$$\begin{aligned} x(t) &= Re[r\cos t + \imath(r\sin t)] \\ &= r\cos t \\ y(t) &= Im[r\cos t + \imath(r\sin t)] \\ &= r\sin t \end{aligned}$$

where r is constant. To map a ray emanating from the origin we want all the points with a certain angle so we use the following parametrization of $\lambda = x + \imath y$

$$\begin{aligned} x(t) &= Re[t\cos\alpha + \imath(t\sin\alpha)] \\ y(t) &= Im[t\cos\alpha + \imath(t\sin\alpha)] \end{aligned}$$

where α is constant.

The largest bulb which has the shape of a cardioid is the only one with an attracting cycle of period one. Luckily $\mu(\lambda)$ turns out to be simple to solve, even by hand. The result becomes

$$\mu_1(\lambda) = \frac{1}{4}(2\lambda - \lambda^2).$$

If this is not already familiar it should be noted that it resembles the complex mapping $z \mapsto z^2$ which maps the unit disc to a cardioid.

The mapping of the two period bulb can also be easily found as well.

$$\mu_2(\lambda) = \frac{1}{4}\lambda - 1$$

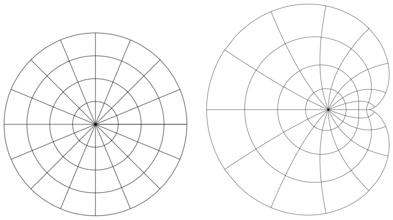


Figure 1.3: Unit disc and mapping of $\mu(\lambda) = c$ from the unit disc to the bulb of period 1.

It should be clear that this maps the unit disc to a disc a quarter of the size and shifted by one to the left; see Figure 1.4.

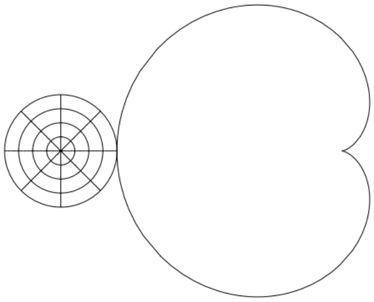


Figure 1.4: Mapping of $\mu(\lambda) = c$ for the bulb of period 2.

Moving on to the three period bulb we can see the solutions get much more complicated. The computer gives us three solutions when solving this case.

$$\begin{aligned}
\mu_{3,1}(\lambda) &= -\frac{2}{3} - \\
&\frac{-\frac{8}{3}-\lambda}{2^{\frac{2}{3}}(-1600+288\lambda-27\lambda^2+3\sqrt{3}\sqrt{94208-34816\lambda+6016\lambda^2-608\lambda^3+27\lambda^4})^{\frac{1}{3}}} \\
&+\frac{(-1600+288\lambda-27\lambda^2+3\sqrt{3}\sqrt{94208-34816\lambda-6016\lambda^2-608\lambda^3+27\lambda^4})^{\frac{1}{3}}}{12(2)^{\frac{1}{3}}} \\
\mu_{3,2}(\lambda) &= -\frac{2}{3} + \\
&\frac{(1+\imath\sqrt{3})(-\frac{8}{3}-\lambda)}{2^{\frac{2}{3}}(-1600+288\lambda-27\lambda^2+3\sqrt{3}\sqrt{94208-34816\lambda+6016\lambda^2-608\lambda^3+27\lambda^4})^{\frac{1}{3}}} \\
&+\frac{(1-\imath\sqrt{3})(-1600+288\lambda-27\lambda^2}{24(2)^{\frac{1}{3}}} \\
&+\frac{(1-\imath\sqrt{3})(3\sqrt{3}\sqrt{94208-34816\lambda-6016\lambda^2-608\lambda^3+27\lambda^4})^{\frac{1}{3}}}{24(2)^{\frac{1}{3}}} \\
\mu_{3,3}(\lambda) &= -\frac{2}{3} + \\
&\frac{(1-\imath\sqrt{3})(-\frac{8}{3}-\lambda)}{2^{\frac{2}{3}}(-1600+288\lambda-27\lambda^2+3\sqrt{3}\sqrt{94208-34816\lambda+6016\lambda^2-608\lambda^3+27\lambda^4})^{\frac{1}{3}}} \\
&+\frac{(1+\imath\sqrt{3})(-1600+288\lambda-27\lambda^2}{24(2)^{\frac{1}{3}}} \\
&+\frac{(1+\imath\sqrt{3})(3\sqrt{3}\sqrt{94208-34816\lambda-6016\lambda^2-608\lambda^3+27\lambda^4})^{\frac{1}{3}}}{24(2)^{\frac{1}{3}}}
\end{aligned}$$

Since there are three bulbs it should make sense that we would get three equations, but as it turns out each of the previous equations map out half of one bulb and half of another. This may explain why the equations do not resemble anything that would map a disc to a disc. Unfortunately there is no sure way of knowing if the bulbs are circular or not unless we could find another way of mapping the edge of each bulb or by checking the curvature.

The image of these three period maps can be seen in figure 1.5. The dot on the left side of the image is actually another Cardioid like the bulb of period one.

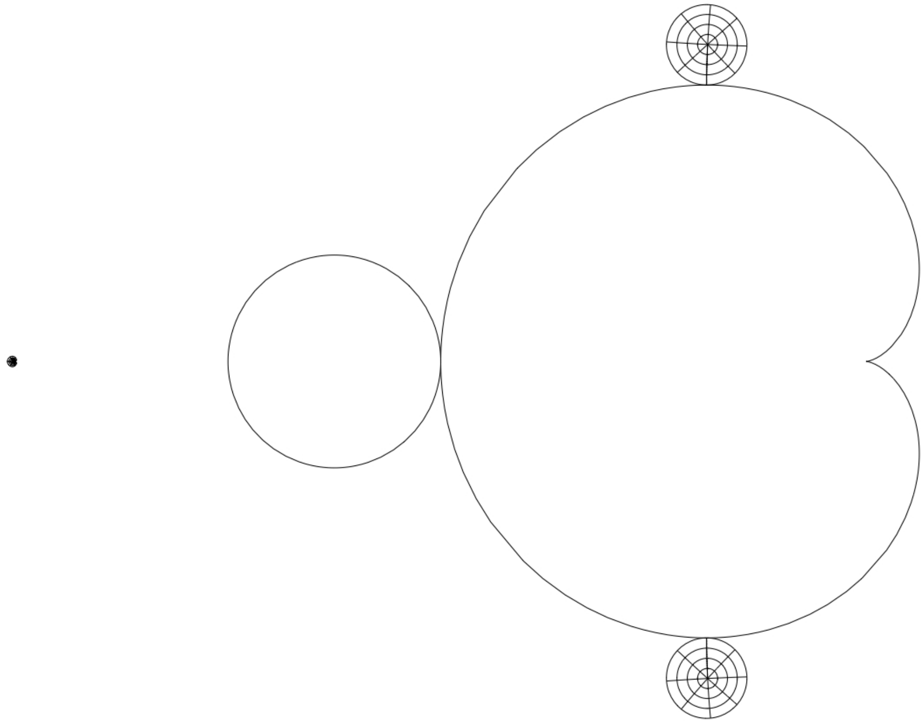


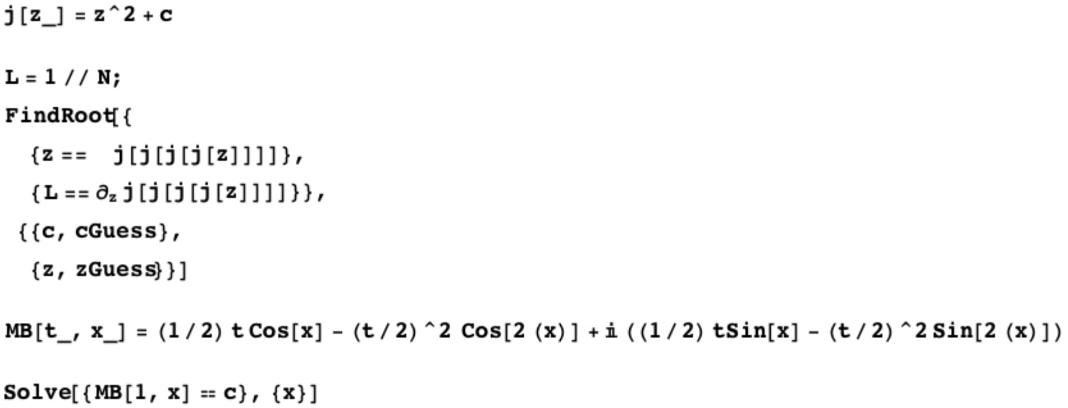
Figure 1.5: Mapping of $\mu(\lambda) = c$ for bulbs of period 3.

Moving up to bulbs where the attracting cycle has a period larger than three we find that most computers are not capable of solving the system of equations to find $\mu(\lambda)$. For these other bulbs another method will be needed.

1.3.2 Point-By-Point Method

To explore bulbs with higher periods we need to use some method other than a computer's brute force to solve a complex system of equations. Instead we can give the computer an easier task of finding the image of each point on a particular bulb of $\mathcal{M}$.

Using an algorithm to find the roots of a function given an initial value similar to Newton's method for finding roots, we can find the c-value which satisfies the system of equations given some λ. From our initial guess of the c-value we are looking for we also need a z-value from the attracting cycle of the corresponding Julia set. This z-value is easily found by iterating the fixed point of the Julia set and watching what it gets attracted to. After a hundred iterations we can be fairly certain the value found is close enough to a point on the attracting cycle. Running the mathematica code below with these two guesses in the variables *cGuess* and *zGuess* will return the actual c and z-values which satisfy the equations. In the next step the c-value found by the *FindRoot* method is used to find the angle of the point which maps to this value from the main cardioid. The code that we use in *Mathematica* is:



```
j[z_] = z^2 + c

L = 1 // N;
FindRoot[{
  {z ==  j[j[j[j[z]]]]},
  {L == ∂z j[j[j[j[z]]]]}},
 {{c, cGuess},
  {z, zGuess}}]

MB[t_, x_] = (1 / 2) t Cos[x] - (t / 2) ^2 Cos[2 (x)] + ⅈ ((1 / 2) tSin[x] - (t / 2) ^2 Sin[2 (x)])

Solve[{MB[1, x] == c}, {x}]
```

Notice that one set of initial c and z-values will allow us to map the entire bulb from which the c-value was taken. Below we can see an example of this process done on the 4-period bulbs. The dots set apart from the other bulbs are also bulbs which are too small to be shown in detail. See Figure 1.6.

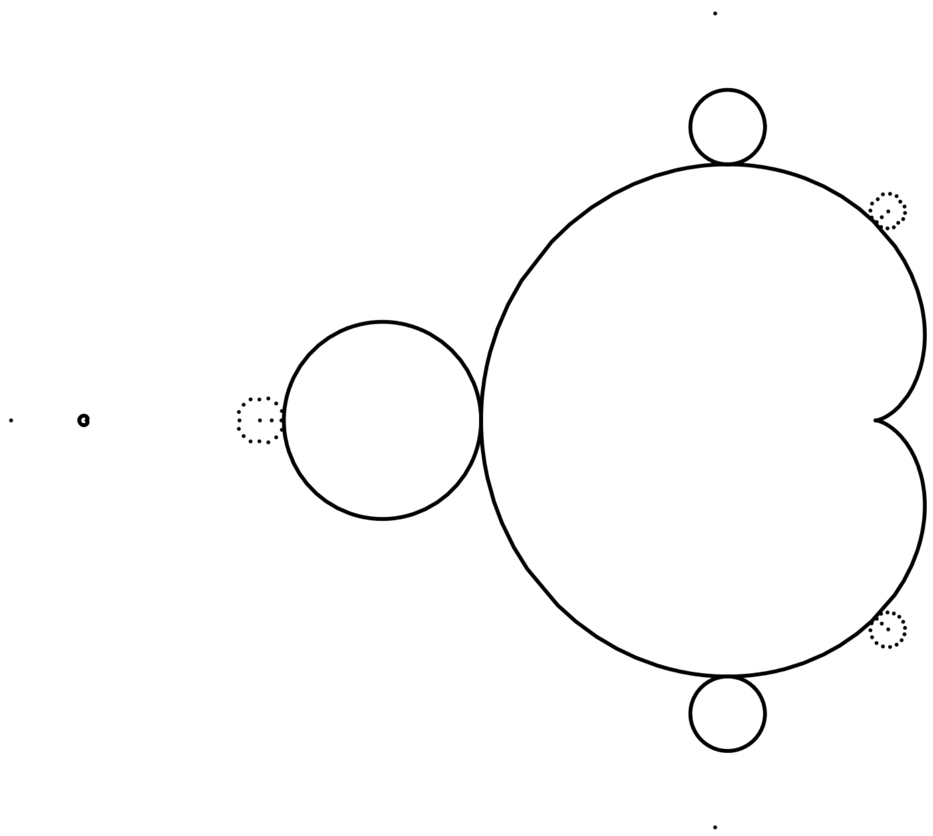


Figure 1.6: Point-by-point estimation of the map $\mu(\lambda) = c$ for bulbs of period 4.

1.4 Mapping The Cubic Parameter Plane

After a method has proven successful for the quadratic, a natural question one might ask is what about higher degree polynomials? See Figure 1.7 for the cubic connectedness locus for the family of functions $C(\lambda) = z^3 + c$. Our system to solve now becomes:

$$\begin{aligned} C_c^n(z) &= z \\ (C_c^n)'(z) &= \lambda \end{aligned}$$

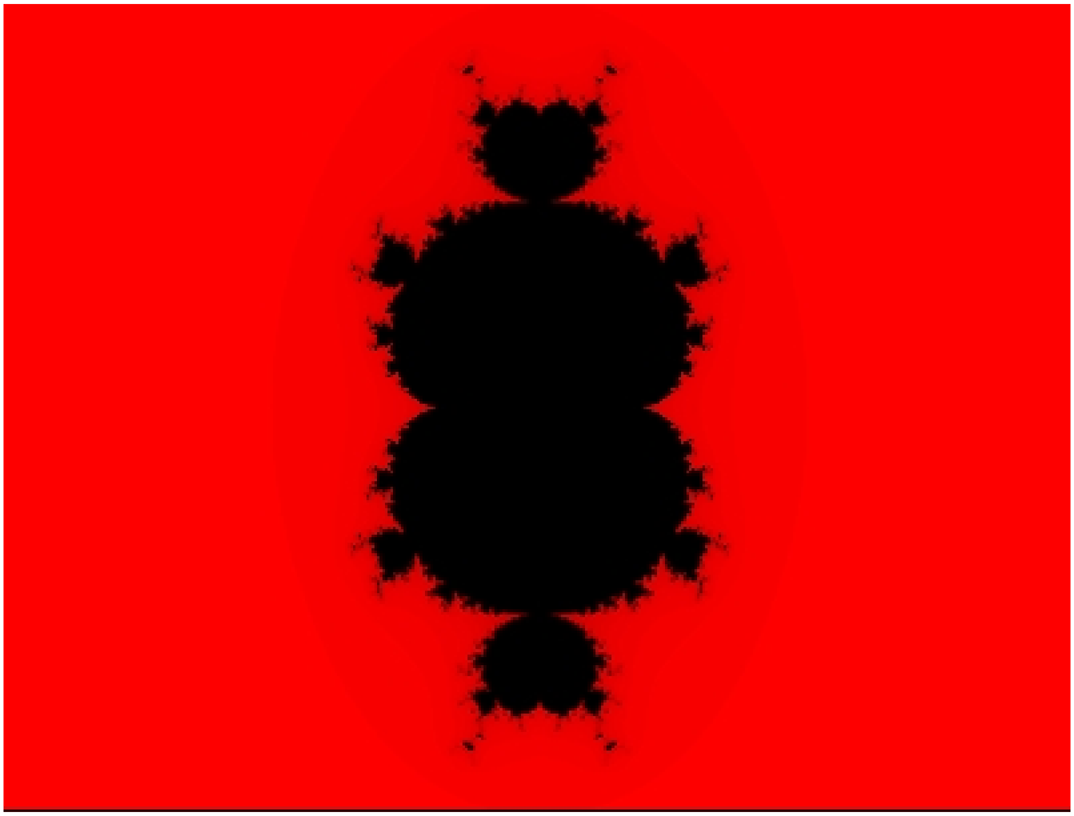


Figure 1.7: The cubic Mandelbrot set.

An important difference between the cubic and quadratic polynomials is that the cubic has two critical points where as the quadratic has only one. This will be more of an issue when, at higher periods, we have to turn to solving for roots to estimate the mapping point-by-point.

The 1 and 2-period bulbs turn out to be solvable by the Mathematica's Solve method. For the main bulb of period 1, the computer returns these two results:

$$\mu_1(\lambda) = \frac{1}{9}(\pm 3\sqrt{3}\sqrt{\lambda} \mp \sqrt{3}\lambda^{\frac{3}{2}})$$

The reason for two equations is because there are two critical points. Below we can see the image of these mappings and also the image of one of the mappings by itself to illustrate how multiple critical points affect the process.

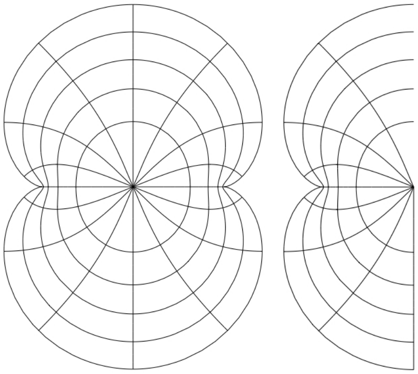


Figure 1.8: Period 1 bulb of the Cubic Mandelbrot and again when only one critical point is considered.

Solving for the period 2 bulbs we should expect four equations since there are two bulbs and two critical points, which is exactly what we get. Unlike the period one bulbs where each critical point gave us a connected area, the period 2 bulbs each have four regions where each non-adjacent pair share the same critical point. This is shown in figure 1.9. The sections of the mapping that share the same critical point are mapped with or without a grid.

Moving on to the bulbs with an attracting cycle of period 3 the computer can no longer solve the system of equations. We must again move to using a point-by-point method of estimating the map. An image of this map for one of the bulbs can be seen in figure 1.10. The difference in color shows how the two regions from the two different critical points overlap.

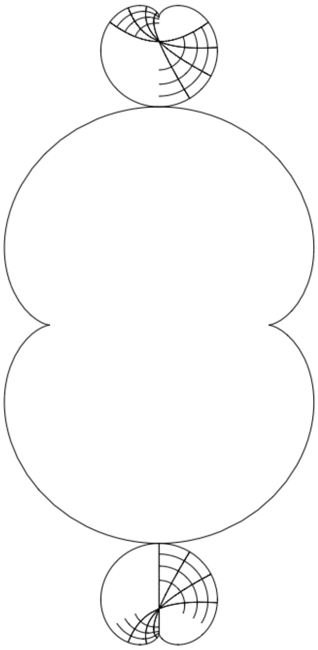


Figure 1.9: Period 2 bulbs of the Cubic Mandelbrot set.

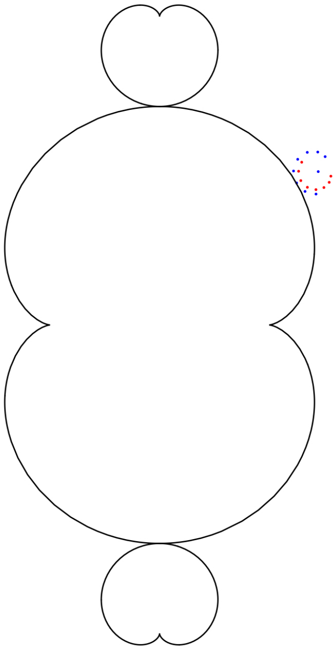


Figure 1.10: An example of a period 3 bulb of the Cubic Mandelbrot set.

CHAPTER 2

INTERNAL RAYS OF THE MANDELBROT SET

In the process of building the images of various bulbs it becomes apparent that every bulb which is attached to a larger bulb is attached at the image of 1 in the smaller bulb (see fig. 1.5). Finding the point whose image in the larger bulb connects the two bulbs turns out to be a difficult problem to do by hand so the use of a computer to do the calculations is essential. By taking the image of 1 in the smaller bulb and setting it equal to the equation that gives us the c-values of the larger bulb we can solve for λ. Since this λ will be on the boundary of the unit disc it will be of the form $\lambda = e^{\imath\theta}$.

More thorough investigation of these relationships between bulbs suggests there is correlation between the angle of the ray whose image attaches to an external bulb and the period of that external bulb.

2.1 The Main Cardioid

Let us first look at how this works in the main cardioid. Below we can see some internal rays and the center points of the external bulbs they connect to. In this case the period of the external bulb is found in the angle as the denominator of the fraction of 2π. For example at the image of $e^{\imath\frac{1}{3}2\pi}$ there will be a bulb with an attracting cycle of period 3.

2.2 Extending the Conjecture to Other Bulbs

We can also notice a correlation between the angles of internal rays and the external bulbs they connect to in higher period bulbs. Below is a diagram of this in the bulb with attracting cycle of period two. The way the angles are represented the external bulb still has a period equal to the denominator of the fraction of 2π, but these ratios are not shown in reduced form. If we were to show the ratios so that the numerator and denominator were relatively prime we would find that the period is always twice the denominator of the reduced fraction. Notice that the period was 1x the denominator of the ratio in the period one bulb and 2x the denominator of the ratio in the period two bulb. The following is a conjecture of a general rule for these relationships.

Conjecture 5. *Let n, p, k be positive integers where $gcd(k, p) = 1$ and $\mu : \mathbb{D} \mapsto W$ where W is a hyperbolic component of $\mathcal{M}$. Given any n-period bulb W_n of $\mathcal{M}$, a bulb W_{np} of period np will be attached at the point $\mu(e^{\imath \frac{k}{p} 2\pi})$.*

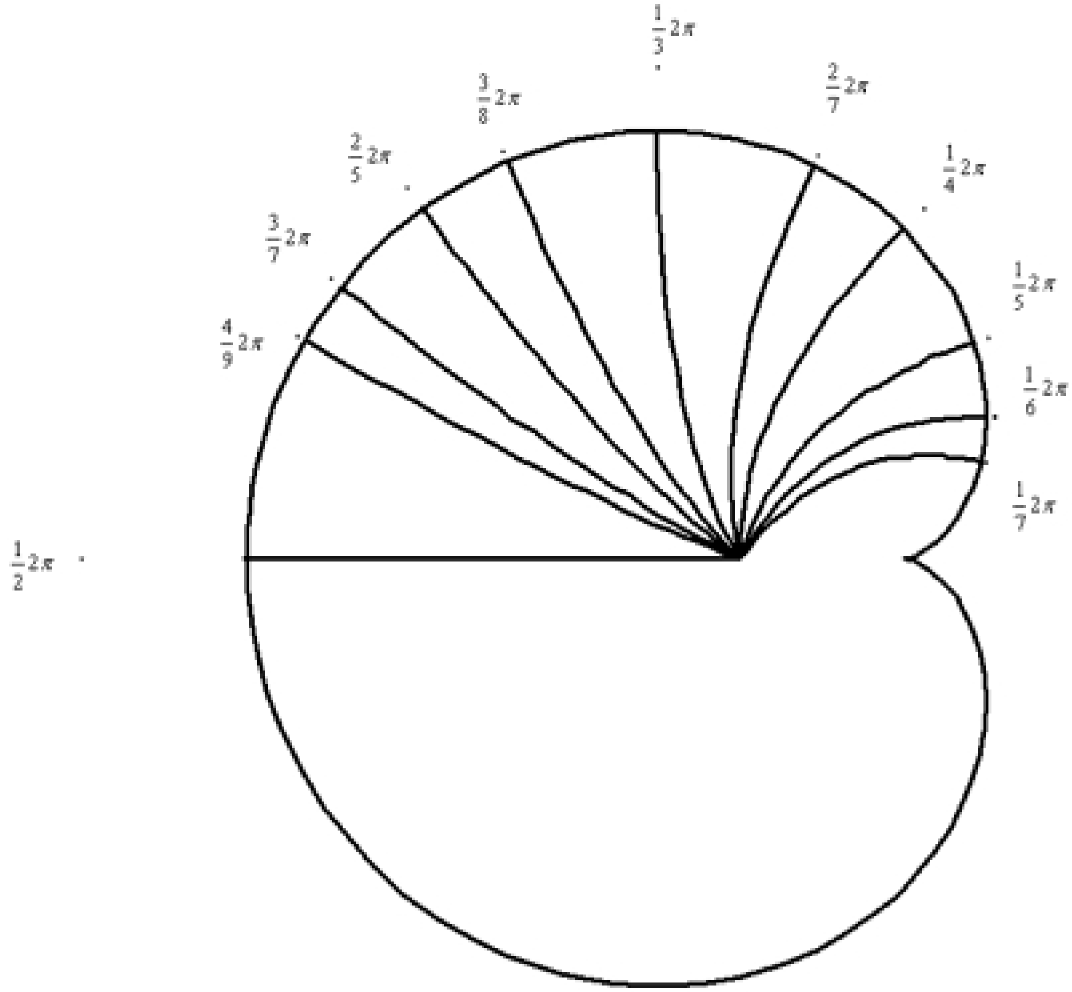


Figure 2.1: Internal rays of the main cardioid and the centers of the external bulbs to which they connect.

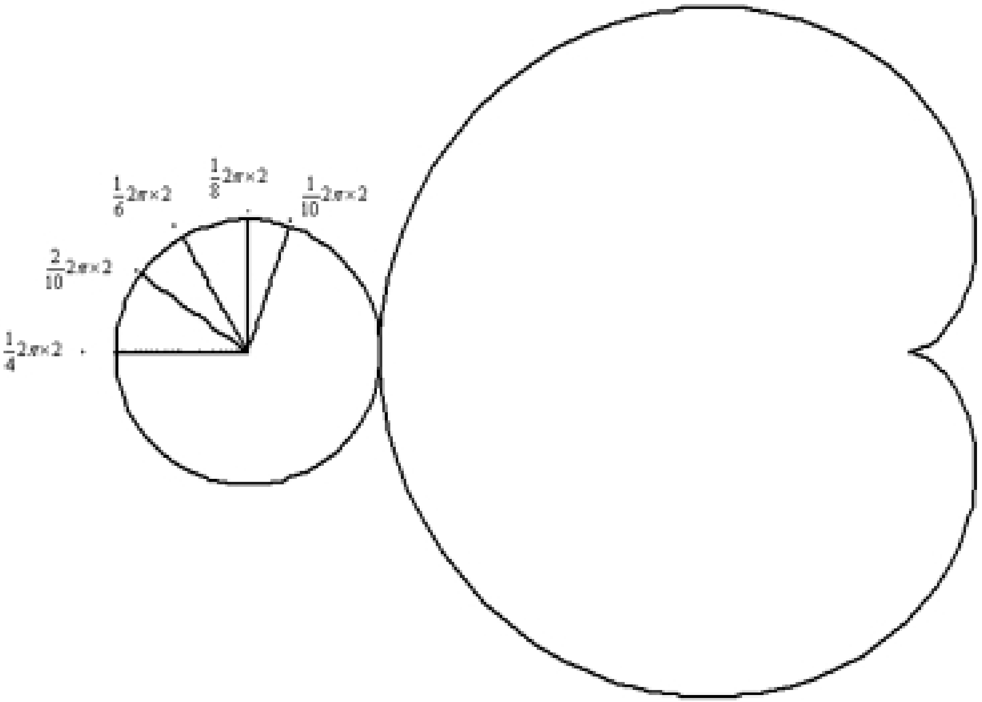


Figure 2.2: Internal rays of the period 2 bulb and the centers of the external bulbs to which they connect.

REFERENCES

[1] Devaney, Robert L. *An Introduction to Chaotic Dynamical Systems*, Second Edition, 1989. Perseus Books Publishing, Reading, MA.

[2] Halsted, Matt. *Dynamics of a Cosine Root Family,* Ithaca College, Mathematics Honors Thesis, 2004.